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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

... ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum. LEIBNIZ

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Über die Existenz kanonischer Transformationen bei mehreren unabhängigen Veränderlichen

HELMUT RÜSSMANN

Vorgelegt von J. L. ERICKSEN

1. Einleitung

Entsprechend der Herleitung der Hamiltonschen Differentialgleichungen aus Variationsproblemen in einer unabhängigen Veränderlichen gelangt man von Variationsproblemen in mehreren unabhängigen Veränderlichen zu Systemen partieller Differentialgleichungen der Gestalt

$$\sum_{i=1}^{n} p_{\alpha i, x_i} = -H_{,q_\alpha}, \qquad q_{\alpha, x_i} = H_{,p_{\alpha i}} \qquad (1)$$
$$(\alpha = 1, 2, \ldots, N;\ i = 1, 2, \ldots, n),$$

wobei $H = H(q_1, \ldots, q_N, p_{11}, \ldots, p_{Nn}, x_1, \ldots, x_n)$ eine in einem gegebenen Gebiet zweimal stetig differenzierbare Funktion ihrer $n + Nn + N$ Argumente ist. Die Indizes hinter dem Komma bedeuten dabei wie üblich partielle Differentiation. Analog zum klassischen Fall $n = 1$ existiert auch für $n > 1$ die Gruppe der „kanonischen" Transformationen, welche jedes System (1) wieder in ein solches überführen. Die notwendige und hinreichende Bedingung dafür, daß eine einmal stetig differenzierbare und umkehrbare Transformation

$$\begin{aligned} q_\alpha &= q_\alpha(Q_1, \ldots, Q_N;\ P_{11}, \ldots, P_{Nn};\ x_1, \ldots, x_n), \\ p_{\alpha i} &= p_{\alpha i}(Q_1, \ldots, Q_N;\ P_{11}, \ldots, P_{Nn};\ x_1, \ldots, x_n) \end{aligned} \qquad (2)$$
$$(\alpha = 1, 2, \ldots, N;\ i = 1, 2, \ldots, n)$$

kanonisch ist, hat D. G. B. EDELEN angegeben[1].

Zur Formulierung dieser Bedingung bezeichne

$$\boldsymbol{M} = (\boldsymbol{M}_{ik}) = \begin{pmatrix} \boldsymbol{M}_{11}, \boldsymbol{M}_{12}, \ldots, \boldsymbol{M}_{1\,n+1} \\ \vdots \\ \boldsymbol{M}_{n+1\,1}, \ldots, \boldsymbol{M}_{n+1\,n+1} \end{pmatrix}$$

die Funktionalmatrix der Transformation (2), wobei also

$$\begin{aligned} \boldsymbol{M}_{11} &= (q_{\alpha, Q_\beta}), & \boldsymbol{M}_{1\,k+1} &= (q_{\alpha, P_{\beta k}}), \\ \boldsymbol{M}_{i+1\,1} &= (p_{\alpha i, Q_\beta}), & \boldsymbol{M}_{i+1\,k+1} &= (p_{\alpha i, P_{\beta k}}) \end{aligned} \qquad (i, k = 1, 2, \ldots, n)$$

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[1] EDELEN, D. G. B.: The Invariance Group for Hamiltonian Systems of Partial Differential Equations I. Analysis. Arch. Rational Mech. Anal. **5** (1960).

Beweis. Aus den Gleichungen (5) erhalten wir durch spezielle Wahl der Indizes für $j>1$, $k>1$

$$\boldsymbol{M}'_{1j}\boldsymbol{M}_{jk}-\boldsymbol{M}'_{jj}\boldsymbol{M}_{1k}=\boldsymbol{O}, \tag{12}$$

$$\boldsymbol{M}'_{11}\boldsymbol{M}_{jk}-\boldsymbol{M}'_{j1}\boldsymbol{M}_{1k}=\frac{1}{\mu}\,\delta_{jk}\boldsymbol{E}, \tag{13}$$

$$\boldsymbol{M}'_{11}\boldsymbol{M}_{j1}-\boldsymbol{M}'_{j1}\boldsymbol{M}_{11}=\boldsymbol{O}. \tag{14}$$

Nach Lemma 1 ist $\det \boldsymbol{M}_{11}\neq 0$. Daher folgt aus (13)

$$\boldsymbol{M}'_{jj}=\frac{1}{\mu}\boldsymbol{M}_{11}^{-1}+\boldsymbol{M}'_{1j}\boldsymbol{M}_{j1}\boldsymbol{M}_{11}^{-1}$$

und aus (13) und (14) für $j\neq k$

$$\boldsymbol{M}_{jk}=\boldsymbol{M}_{j1}\boldsymbol{M}_{11}^{-1}\boldsymbol{M}_{1k}.$$

Setzen wir die beiden Ausdrücke für $\boldsymbol{M}'_{jj}$ und $\boldsymbol{M}_{jk}$ in (12) ein, so ergibt sich $\frac{1}{\mu}\boldsymbol{M}_{11}^{-1}\boldsymbol{M}_{1k}=\boldsymbol{O}$, also

$$\boldsymbol{M}_{1k}=\boldsymbol{O}\qquad(k>1), \tag{15}$$

q.e.d.

Lemma 1 und Lemma 2 zusammen bilden gerade die Aussage des in der Einleitung angekündigten Satzes.

3. Die explizite Darstellung der Lösungen von (3) und der kanonischen Transformationen

Um sämtliche Lösungen des Systems (5) zu bestimmen, beachten wir zunächst (13). Wegen (15) ergibt sich

$$\boldsymbol{M}_{i+1\,k+1}=\frac{1}{\mu}\,\delta_{ik}\boldsymbol{M}'^{-1}_{11}\qquad(i,k=1,2,\ldots,n). \tag{16}$$

Setzen wir noch

$$\boldsymbol{M}_{i+1\,1}=\boldsymbol{M}'^{-1}_{11}\boldsymbol{S}_i\qquad(i=1,2,\ldots,n), \tag{17}$$

so folgt aus (14) $\boldsymbol{S}'_i=\boldsymbol{S}_i$.

Die Gleichungen (15), (16) und (17) mit symmetrischen Matrizen $\boldsymbol{S}_i$ sind also notwendig für das Bestehen der Gleichungen (5). Daß sie auch hinreichen, verifiziert man durch Einsetzen.

Die allgemeinste Lösung des Gleichungssystems (3) lautet also für $n>1$

$$\boldsymbol{M}=\left(\begin{array}{c|ccc}\boldsymbol{M}_{11} & \boldsymbol{O} & \ldots & \boldsymbol{O}\\ \hline \boldsymbol{M}'^{-1}_{11}\boldsymbol{S}_1 & \frac{1}{\mu}\boldsymbol{M}'^{-1}_{11} & & \boldsymbol{O}\\ \vdots & & \ddots & \\ \boldsymbol{M}'^{-1}_{11}\boldsymbol{S}_n & \boldsymbol{O} & & \frac{1}{\mu}\boldsymbol{M}'^{-1}_{11}\end{array}\right) \tag{18}$$

mit beliebiger nicht ausgearteter Matrix $\boldsymbol{M}_{11}$ und beliebigen symmetrischen Matrizen $\boldsymbol{S}_i$.

Zum Schluß wollen wir noch alle kanonischen Transformationen für $n > 1$ bestimmen. Seien hierzu

$$q_\alpha = q_\alpha(Q_1, \ldots, Q_N;\ x_1, \ldots, x_n) \qquad (\alpha = 1, 2, \ldots, N) \tag{19}$$

in einem gewissen Gebiet ihrer $N + n$ Argumente stetig differenzierbare Funktionen mit $\det(q_{\alpha, Q_\beta}) \neq 0$. Wir bezeichnen die Spaltenvektoren mit den Komponenten $P_{1i}, \ldots, P_{Ni}$ mit $\boldsymbol{P}_i$ und erhalten aus (18)

$$\boldsymbol{p}_i = \boldsymbol{M}_{11}'^{-1}\left(\frac{1}{\mu}\,\boldsymbol{P}_i + \boldsymbol{a}_i\right) \qquad (i = 1, 2, \ldots, n), \tag{20}$$

wobei die Komponenten $a_{\alpha i}$ der Spaltenvektoren $\boldsymbol{a}_i$ stetig differenzierbare Funktionen in $Q_1, \ldots, Q_N, x_1, \ldots, x_n$ sind. Differentiation von (20) nach Q_β und Multiplikation mit M_{11}' ergibt dann

$$\begin{aligned} \boldsymbol{M}_{11}'\,\boldsymbol{p}_{i, Q_\beta} &= \boldsymbol{M}_{11}'\,\boldsymbol{M}_{11, Q_\beta}'^{-1}\left(\frac{1}{\mu}\,\boldsymbol{P}_i + \boldsymbol{a}_i\right) + \boldsymbol{a}_{i, Q_\beta} \\ &= \boldsymbol{M}_{11, Q_\beta}'\,\boldsymbol{b}_i + \boldsymbol{a}_{i, Q_\beta}, \end{aligned}$$

wobei $\boldsymbol{b}_i = -\boldsymbol{M}_{11}'^{-1}\left(\frac{1}{\mu}\boldsymbol{P}_i + \boldsymbol{a}_i\right)$ ist. Hieraus folgt

$$\boldsymbol{S}_i = \boldsymbol{M}_{11}'\,\boldsymbol{M}_{i+1\,1} = \left(\sum_\nu b_{\nu i}\, q_{\nu, Q_\alpha, Q_\beta}\right) + (a_{\alpha i, Q_\beta})\,.$$

Die erste Matrix in der Summe ist symmetrisch. Die Symmetrie der zweiten ist gleichbedeutend mit der Existenz von in den Q_α zweimal und in den x_i einmal stetig differenzierbaren Funktionen c_i, so daß

$$a_{\alpha i} = c_{i, Q_\alpha} \tag{21}$$

gilt. Durch (19), (20) und (21) werden also alle kanonischen Transformationen für $n > 1$ dargestellt. Diese Formeln finden sich auch bei EDELEN[4], doch geht aus der dort gegebenen Ableitung nicht hervor, daß keine weiteren kanonischen Transformationen existieren.

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[4] Siehe [1] S. 154.

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Surface Waves in Deformed Elastic Materials

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1. Introduction

In the present paper we apply the theory [*1*][*] of the superposition of infinitesimal deformations on finite deformations in an isotropic elastic material to the study of the propagation of surface waves in a semi-infinite body which is subjected to a static, pure homogeneous deformation.

Waves of the type discovered by RAYLEIGH [*2*] are first examined. It is seen that the motion does not differ essentially from that formulated by RAYLEIGH on the basis of classical elasticity theory. There is no dispersion, as might be expected, since no fundamental length is involved either in the dimensions of the body or in the deformations it undergoes. Also, the motion is two-dimensional in the sense that there is no displacement transverse to the direction of propagation of the waves. A form of strain-energy function, equivalent to that used by MURNAGHAN [*3*], is employed in obtaining the correction, due to an initial small pure homogeneous deformation, in the frequency of propagation of waves of this type. The method, outlined by RIVLIN [*4*], of deducing the results for an incompressible body in second-order elasticity theory, from the corresponding second-order results for a compressible one, is used to obtain in explicit form the displacement components for the case of an incompressible body.

The possibility of having surface waves with transverse horizontal movement in a semi-infinite body, subjected to a pure homogeneous deformation, is next discussed. We find that such waves may not be propagated in a principal direction unless stringent conditions are imposed on the strain-energy function. Then, following LOVE [*5*], we assume that there rests on the semi-infinite body a layer of material with properties differing from those of the semi-infinite body and subjected to a different pure homogeneous deformation. The principal directions are taken to be the same in the layer as in the subjacent body. An infinitesimal deformation is superimposed on the composite body and conditions for the existence of Love-type waves are sought. The conditions obtained are similar to those found by LOVE.

2. The Equations of Motion and Boundary Conditions

We take the origin of a rectangular cartesian coordinate system (X_i) on the surface of a semi-infinite body which occupies the region $X_2 \geqq 0$. This body is subjected to a pure homogeneous deformation defined by[**]

$$x_\alpha = \lambda_\alpha X_\alpha . \tag{2.1}$$

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[*] Numbers in square brackets refer to the references at the end of the paper.

[**] The summation convention will not apply when Greek subscripts are employed.

The stresses σ_{ij} associated with this deformation are given [6] by

$$\begin{aligned} \sigma_{\alpha\beta} &= 0 \qquad (\alpha \neq \beta), \\ \sigma_{\alpha\alpha} &= 2 I_3^{-\frac{1}{2}} \{\lambda_\alpha^2 W_1 + \lambda_\alpha^2 (I_1 - \lambda_\alpha^2) W_2 + I_3 W_3\}, \end{aligned} \tag{2.2}$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, \quad I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2, \quad W_\alpha = \partial W / \partial I_\alpha,$$

W being the strain-energy function.

Finally, we superimpose on the deformation (2.1) an infinitesimal deformation such that the particle whose position is x_i is displaced to a point $\bar{x}_i$, where

$$\bar{x}_i = x_i + \varepsilon u_i. \tag{2.3}$$

Here ε is a constant such that its squares and higher powers may be neglected in comparison with unity. The equations of motion are found to be [7]

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} = \varrho \frac{\partial^2 u_i}{\partial t^2}, \tag{2.4}$$

where $\bar{\sigma}_{ij}$, the incremental stress associated with (2.3), is given by

$$\begin{aligned} \bar{\sigma}_{\alpha\beta} &= 2 I_3^{-\frac{1}{2}} \{W_1 + (I_1 - \lambda_\alpha^2 - \lambda_\beta^2) W_2\} (\lambda_\alpha^2 u_{\beta,\alpha} + \lambda_\beta^2 u_{\alpha,\beta}) \qquad (\alpha \neq \beta), \\ \bar{\sigma}_{\alpha\alpha} &= \sum_{\beta=1}^{3} c_{\alpha\beta} u_{\beta,\beta}, \end{aligned} \tag{2.5}$$

and ϱ is the density of the material in the state of pure homogeneous deformation. Differentiation with respect to x_α is denoted by $_{,\alpha}$ and $c_{\alpha\beta}$ is given by

$$\begin{aligned} c_{\alpha\beta} = 2 I_3^{-\frac{1}{2}} \{ & [W_1 + (I_1 - \lambda_\alpha^2 - \lambda_\beta^2) W_2] \lambda_\alpha^2 (2\delta_{\alpha\beta} - 1) + \lambda_\alpha^2 \lambda_\beta^2 W_2 + I_3 W_3 + \\ & + 2\{\lambda_\alpha^2 [W_{11} + (I_1 - \lambda_\alpha^2) W_{21}] + I_3 W_{31}\} \lambda_\beta^2 + 2\{\lambda_\alpha^2 [W_{12} + (I_1 - \lambda_\alpha^2) W_{22}] + \\ & + I_3 W_{32}\} \lambda_\beta^2 (I_1 - \lambda_\beta^2) + 2\{\lambda_\alpha^2 [W_{13} + (I_1 - \lambda_\alpha^2) W_{23}] + I_3 W_{33}\} I_3\}, \end{aligned} \tag{2.6}$$

where $\partial^2 W / \partial I_\alpha \partial I_\beta$ is denoted by $W_{\alpha\beta}$.

The surface tractions T_i, in the state of deformation (2.1), are given by

$$T_i = \sigma_{ij} l_j, \tag{2.7}$$

where l_j are the direction-cosines of the normal to the surface in that state. The surface tractions, $T_i + \varepsilon \bar{T}_i$, in the state of deformation (2.3) are given by

$$T_i + \varepsilon \bar{T}_i = (\sigma_{ij} + \varepsilon \bar{\sigma}_{ij}) (l_j + \varepsilon \bar{l}_j), \tag{2.8}$$

where $\sigma_{ij} + \varepsilon \bar{\sigma}_{ij}$, $l_j + \varepsilon \bar{l}_j$ are the stress components and direction-cosines of the normal to the surface in that state. It has been shown [6] that

$$\bar{l}_j = \tfrac{1}{2} l_j l_m l_n (u_{m,n} + u_{n,m}) - l_k u_{k,j}. \tag{2.9}$$

Neglecting powers of ε higher than the first and using (2.7), (2.8) reduces to

$$\bar{T}_i = \sigma_{ij} \bar{l}_j + \bar{\sigma}_{ij} l_j. \tag{2.10}$$

3. The Frequency Equation

We assume the surface $x_2=0$ to be disturbed in some manner and suppose the displacement components u_i to be given by

$$\begin{aligned} u_1 &= f_1(x_2)\exp\iota(p\,x_1+q\,t), \\ u_2 &= f_2(x_2)\exp\iota(p\,x_1+q\,t), \\ u_3 &= 0, \end{aligned} \tag{3.1}$$

where f_1 and f_2 are to be determined. Substituting these values in (2.5) and using (2.4), we find that

$$\begin{aligned} &\{(\varrho\,q^2-c_{11}\,p^2)+\lambda_2^2\,b\,D^2\}\,f_1+(c_{12}+\lambda_1^2\,b)\,\iota\,p\,D\,f_2=0, \\ &\{(\varrho\,q^2-\lambda_1^2\,b\,p^2)+c_{22}\,D^2\}\,f_2+(c_{21}+\lambda_2^2\,b)\,\iota\,p\,D\,f_1=0, \end{aligned} \tag{3.2}$$

where

$$D=d/d\,x_2, \qquad b=2I_3^{-\frac{1}{2}}\{W_1+(I_1-\lambda_1^2-\lambda_2^2)\,W_2\}.$$

The third equation of motion is satisfied identically. To solve equations (3.2), we write

$$\begin{aligned} f_1 &= -\,(c_{12}+\lambda_1^2\,b)\,\iota\,p\,D\,V, \\ f_2 &= (\varrho\,q^2-c_{11}\,p^2+\lambda_2^2\,b\,D^2)\,V, \end{aligned} \tag{3.3}$$

where $V(x_2)$ is to be determined. Evidently, these values of f_1, f_2 satisfy the first of equations (3.2) whatever may be the value of V. Substituting them in the second of (3.2) we find

$$\begin{aligned} &\{\lambda_2^2\,b\,c_{22}\,D^4+[c_{22}(\varrho\,q^2-c_{11}\,p^2)+\lambda_2^2\,b\,(\varrho\,q^2-\lambda_1^2\,b\,p^2)+ \\ &\quad+(c_{12}+\lambda_1^2\,b)\,(c_{21}+\lambda_2^2\,b)\,p^2]\,D^2+(\varrho\,q^2-c_{11}\,p^2)\,(\varrho\,q^2-\lambda_1^2\,b\,p^2)\}\,V=0. \end{aligned} \tag{3.4}$$

The general solution of (3.4) is

$$V=A\,e^{-m_1x_2}+B\,e^{-m_2x_2}+A_1\,e^{m_1x_2}+B_1\,e^{m_2x_2}, \tag{3.5}$$

where A, B, A_1, B_1 are constants and $\pm m_1$, $\pm m_2$ are the roots of the equation

$$\begin{aligned} &\lambda_2^2\,b\,c_{22}\,m^4+\{c_{22}(\varrho\,q^2-c_{11}\,p^2)+\lambda_2^2\,b\,(\varrho\,q^2-\lambda_1^2\,b\,p^2)+ \\ &\quad+(c_{12}+\lambda_1^2\,b)\,(c_{21}+\lambda_2^2\,b)\,p^2\}\,m^2+(\varrho\,q^2-c_{11}\,p^2)\,(\varrho\,q^2-\lambda_1^2\,b\,p^2)=0. \end{aligned} \tag{3.6}$$

We may now find u_1, u_2 from (3.1), (3.3) and (3.5). If u_1, u_2 are to tend to zero as x_2 tends to infinity, then, taking $m_1, m_2>0$, we must have

$$A_1=B_1=0.$$

Then

$$\begin{aligned} u_1 &= (c_{12}+\lambda_1^2\,b)\,\iota\,p\,(m_1\,A\,e^{-m_1x_2}+m_2\,B\,e^{-m_2x_2})\exp\iota(p\,x_1+q\,t), \\ u_2 &= \{(\varrho\,q^2-c_{11}\,p^2+m_1^2\,\lambda_2^2\,b)\,A\,e^{-m_1x_2} \\ &\quad+(\varrho\,q^2-c_{11}\,p^2+m_2^2\,\lambda_2^2\,b)\,B\,e^{-m_2x_2}\}\exp\iota(p\,x_1+q\,t). \end{aligned} \tag{3.7}$$

These displacement components correspond to waves, propagated parallel to the surface of the body, with their amplitudes decreasing with distance from the surface. They are analogous to Rayleigh surface waves in the classical theory (see, for example, [2]).

We shall now specify the boundary conditions to be satisfied on the surface of the body and from them determine an equation for the ratio $p^2/\varrho q^2$. The values of $\bar{l}_i$ on the surface may be found from (2.9) by replacing l_1, l_2, l_3 by $0, -1, 0$ and taking note of the fact that u_1, u_2 are not functions of x_3. We find $\bar{l}_1 = u_{2,1}$, $\bar{l}_2 = \bar{l}_3 = 0$. Thus, if we take the surface to be free of traction both in the state of pure homogeneous deformation and in the state of the superimposed infinitesimal deformation, the boundary conditions

$$\sigma_{22} = 0, \qquad \sigma_{11} u_{2,1} - \bar{\sigma}_{12} = 0, \qquad \bar{\sigma}_{22} = 0, \qquad \bar{\sigma}_{32} = 0, \tag{3.8}$$

must be satisfied on the surface $x_2 = 0$. The last of these conditions is satisfied identically. The first condition gives

$$W_1 + (I_1 - \lambda_2^2) W_2 + \lambda_1^2 \lambda_3^2 W_3 = 0. \tag{3.9}$$

The second condition gives, using (2.2) and (2.5),

$$\begin{aligned}\{\lambda_1^2 W_1 + \lambda_1^2 (I_1 - \lambda_1^2) W_2 + I_3 W_3\} u_{2,1} - {}\\ - \{W_1 + (I_1 - \lambda_1^2 - \lambda_2^2) W_2\} (\lambda_1^2 u_{2,1} + \lambda_2^2 u_{1,2}) = 0.\end{aligned} \tag{3.10}$$

With (3.9), this may be written

$$\lambda_1^2 \lambda_2^2 (W_2 + \lambda_3^2 W_3)(u_{1,2} + u_{2,1}) = 0. \tag{3.11}$$

Thus, provided $\lambda_1^2 \lambda_2^2 (W_2 + \lambda_3^2 W_3) \neq 0$, we have

$$u_{1,2} + u_{2,1} = 0. \tag{3.12}$$

The third boundary condition (3.8) gives, using (2.5),

$$c_{21} u_{1,1} + c_{22} u_{2,2} = 0. \tag{3.13}$$

Substituting u_1, u_2 from (3.7) into (3.12), we find

$$\begin{aligned}p\,[\{(\varrho q^2 - c_{11} p^2) + m_1^2 \{(\lambda_2^2 - \lambda_1^2) b - c_{12}\}\} A + {}\\ + \{(\varrho q^2 - c_{11} p^2) + m_2^2 \{(\lambda_2^2 - \lambda_1^2) b - c_{12}\}\} B] = 0.\end{aligned} \tag{3.14}$$

Thus we have either

$$p = 0, \tag{3.15}$$

or

$$\begin{aligned}\{(\varrho q^2 - c_{11} p^2) + m_1^2 [(\lambda_2^2 - \lambda_1^2) b - c_{12}]\} A + {}\\ + \{(\varrho q^2 - c_{11} p^2) + m_2^2 [(\lambda_2^2 - \lambda_1^2) b - c_{12}]\} B = 0.\end{aligned} \tag{3.16}$$

From (3.13), again using (3.7), we find

$$\begin{aligned}m_1 [(c_{12} c_{21} - c_{11} c_{22} + c_{21} \lambda_1^2 b) p^2 + c_{22} \varrho q^2 + c_{22} \lambda_2^2 b m_1^2] A + {}\\ + m_2 [(c_{12} c_{21} - c_{11} c_{22} + c_{21} \lambda_1^2 b) p^2 + c_{22} \varrho q^2 + c_{22} \lambda_2^2 b m_2^2] B = 0.\end{aligned} \tag{3.17}$$

In order that (3.16) and (3.17) have non-vanishing solutions for the integration constants A, B we must have

$$\begin{vmatrix} (\varrho q^2 - c_{11} p^2) + m_1^2 \{(\lambda_2^2 - \lambda_1^2) b - c_{12}\}, \\ \qquad m_1 \{(c_{12} c_{21} - c_{11} c_{22} + c_{21} \lambda_1^2 b) p^2 + c_{22} \varrho q^2 + c_{22} \lambda_2^2 b m_1^2\} \\ (\varrho q^2 - c_{11} p^2) + m_2^2 \{(\lambda_2^2 - \lambda_1^2) b - c_{12}\}, \\ \qquad m_2 \{(c_{12} c_{21} - c_{11} c_{22} + c_{21} \lambda_1^2 b) p^2 + c_{22} \varrho q^2 + c_{22} \lambda_2^2 b m_2^2\} \end{vmatrix} = 0. \tag{3.18}$$

This equation may be written

$$\begin{aligned}(m_1 - m_2)\{(\varrho q^2 - c_{11} p^2)\,[(c_{12} c_{21} - c_{11} c_{22} + c_{21} \lambda_1^2 b)\, p^2 + c_{22} \varrho q^2] + {}\\ + c_{22} \lambda_2^2 b\, (\varrho q^2 - c_{11} p^2)\,(m_1^2 + m_2^2) - c_{22} \lambda_2^2 b\, [c_{12} + (\lambda_1^2 - \lambda_2^2)\, b]\, m_1^2 m_2^2 - {}\\ - m_1 m_2 (c_{12} + \lambda_1^2 b)\,[(c_{12} c_{21} - c_{11} c_{22} + c_{21} (\lambda_1^2 - \lambda_2^2)\, b)\, p^2 + c_{22} \varrho q^2]\} = 0.\end{aligned} \tag{3.19}$$

Inserting the values of $m_1^2 + m_2^2$, $m_1^2 m_2^2$ obtainable from (3.6), in (3.19), we find either

$$m_1 = m_2, \tag{3.20}$$

or

$$\begin{aligned}(c_{12} + \lambda_1^2 b)\,(\varrho q^2 - c_{11} p^2)\,\{\varrho q^2 - (\lambda_1^2 - \lambda_2^2)\, b\, p^2\}\\ = (c_{12} + \lambda_1^2 b)\,\{[c_{11} c_{22} - c_{12} c_{21} - c_{21} (\lambda_1^2 - \lambda_2^2)\, b]\, p^2 - c_{22} \varrho q^2\}\, m_1 m_2.\end{aligned} \tag{3.21}$$

Assuming $c_{12} + \lambda_1^2 b \neq 0$, squaring both sides of (3.21) and again using (3.6), we have

$$\begin{aligned}(\varrho q^2 - c_{11} p^2)^2 \{\varrho q^2 - (\lambda_1^2 - \lambda_2^2)\, b\, p^2\}^2 \lambda_2^2 b\, c_{22}\\ = (\varrho q^2 - c_{11} p^2)\,(\varrho q^2 - \lambda_1^2 b\, p^2)\,\{[c_{11} c_{22} - c_{12} c_{21} - c_{21} (\lambda_1^2 - \lambda_2^2)\, b]\, p^2 - c_{22} \varrho q^2\}^2.\end{aligned} \tag{3.22}$$

Thus either

$$\varrho q^2 = c_{11} p^2, \tag{3.23}$$

or

$$\begin{aligned}(\varrho q^2 - c_{11} p^2)\,\{\varrho q^2 - (\lambda_1^2 - \lambda_2^2)\, b\, p^2\}^2 \lambda_2^2 b\, c_{22}\\ = (\varrho q^2 - \lambda_1^2 b\, p^2)\,\{[c_{11} c_{22} - c_{12} c_{21} - c_{21} (\lambda_1^2 - \lambda_2^2)\, b]\, p^2 - c_{22} \varrho q^2\}^2.\end{aligned} \tag{3.24}$$

Writing

$$p^2/\varrho q^2 = \alpha, \tag{3.25}$$

(3.24) may be written

$$\begin{aligned}(1 - c_{11} \alpha)\,[1 - (\lambda_1^2 - \lambda_2^2)\, b\, \alpha]^2 \lambda_2^2 b\, c_{22}\\ = (1 - \lambda_1^2 b\, \alpha)\,\{[c_{11} c_{22} - c_{12} c_{21} - c_{21} (\lambda_1^2 - \lambda_2^2)\, b]\, \alpha - c_{22}\}^2.\end{aligned} \tag{3.26}$$

In the next section we shall show that the solutions given by (3.15), (3.20), (3.23) are degenerate. We refer to (3.26) as the *frequency equation* since it governs the frequency with which Rayleigh surface waves of given wave-length are propagated. We note that, since the coefficients of α in (3.26) do not involve either p or q, there is no dispersion, as indeed is expected since there is no fundamental length involved either in the dimensions of the body or in the deformation it undergoes.

4. The degenerate solutions

In this section we examine the solutions (3.15), (3.20), (3.23) and the case when $c_{12} + \lambda_1^2 b = 0$.

If $p = 0$ (*cf.* (3.15)) then the equations (3.2) become

$$\begin{aligned}(\varrho q^2 + \lambda_2^2 b\, D^2)\, f_1 = 0,\\ (\varrho q^2 + c_{22} D^2)\, f_2 = 0.\end{aligned} \tag{4.1}$$

Solving for f_1 and f_2 and substituting the expressions obtained in (3.1), we find for u_1 and u_2,

$$\begin{aligned} u_1 &= [A \cos(\varrho\, q^2/\lambda_2^2\, b)^{\frac{1}{2}}\, x_2 + B \sin(\varrho\, q^2/\lambda_2^2\, b)^{\frac{1}{2}}\, x_2] \exp \iota\, q\, t, \\ u_2 &= [A_1 \cos(\varrho\, q^2/c_{22})^{\frac{1}{2}}\, x_2 + B_1 \sin(\varrho\, q^2/c_{22})^{\frac{1}{2}}\, x_2] \exp \iota\, q\, t, \end{aligned} \tag{4.2}$$

where A, B, A_1 and B_1 are constants. In order that the boundary condition (3.12) be satisfied we must have either $q=0$, or $B=0$. In the former case we do not have a wave. We conclude that $B=0$. In similar fashion, from the condition (3.13) we conclude that $B_1=0$. Thus we have

$$\begin{aligned} u_1 &= A \cos(\varrho\, q^2/\lambda_2^2\, b)^{\frac{1}{2}}\, x_2 \exp \iota\, q\, t, \\ u_2 &= A_1 \cos(\varrho\, q^2/c_{22})^{\frac{1}{2}}\, x_2 \exp \iota\, q\, t, \end{aligned} \tag{4.3}$$

where A and A_1 are arbitrary constants. The expression for u_1 represents a standing wave in the x_3-direction, polarized in the x_1-direction, with wave-length $2\pi(\lambda_2^2 b/\varrho\, q^2)^{\frac{1}{2}}$ and angular frequency q. The expression for u_2 represents a standing wave in the x_3-direction, polarized in the x_2-direction, with wave-length $2\pi(c_{22}/\varrho\, q^2)^{\frac{1}{2}}$ and angular frequency q.

Again, if $m_1=m_2$ (*cf.* (3.20)), from (3.6) we have

$$\begin{aligned} &4\lambda_2^2\, b\, c_{22}(\varrho\, q^2 - c_{11}\, p^2)\,(\varrho\, q^2 - \lambda_1^2\, b\, p^2) \\ &\quad = \{c_{22}(\varrho\, q^2 - c_{11}\, p^2) + \lambda_2^2\, b\,(\varrho\, q^2 - \lambda_1^2\, b\, p^2) + (c_{12} + \lambda_1^2\, b)\,(c_{21} + \lambda_2^2\, b)\, p^2\}^2. \end{aligned} \tag{4.4}$$

In this case the general solution for V is

$$V = (A + B\, x_2)\, e^{-m x_2} + (A_1 + B_1\, x_2)\, e^{m x_2}, \tag{4.5}$$

where A, B, A_1 and B_1 are constants and m is the positive root of (3.6). We may find the displacement components u_1, u_2 from (3.1) and (3.3). In order that u_1 and u_2 tend to zero as x_2 tends to infinity we must have $A_1=B_1=0$. Then

$$\begin{aligned} u_1 &= (c_{12} + \lambda_1^2\, b)\, \iota\, p\,\{m(A + B\, x_2) - B\}\, e^{-m x_2} \exp \iota(p\, x_1 + q\, t), \\ u_2 &= \{(\varrho\, q^2 - c_{11}\, p^2 + \lambda_2^2\, b\, m^2)\,(A + B\, x_2) - 2\lambda_2^2\, b\, m\, B\}\, e^{-m x_2} \exp \iota(p\, x_1 + q\, t). \end{aligned} \tag{4.6}$$

If we substitute these values of u_1 and u_2 in the boundary conditions (3.12) and (3.13), we find two equations replacing (3.16) and (3.17). In order that these equations have non-trivial solutions for the constants A and B, we must have

$$\begin{vmatrix} p\{(\varrho\, q^2 - c_{11}\, p^2) + [(\lambda_2^2 - \lambda_1^2)\, b - c_{12}]\, m^2\}, & 2p\, m[(\lambda_1^2 - \lambda_2^2)\, b + c_{12}] \\ -m\{c_{21}(c_{12} + \lambda_1^2\, b)\, p^2 + c_{22}(\varrho\, q^2 - c_{11}\, p^2 + \lambda_2^2\, b\, m^2)\}, & \{c_{21}(c_{12} + \lambda_1^2\, b)\, p^2 + c_{22}(\varrho\, q^2 - c_{11}\, p^2 + 3\lambda_2^2\, b\, m^2)\} \end{vmatrix} = 0. \tag{4.7}$$

If we expand this determinant and, bearing in mind the fact that $m_1=m_2$, substitute the values of m^2 and m^4 from (3.6), we find an equation for p^2 in terms of q^2. However, the resulting equation is different from (4.4). Accordingly we conclude that we do not have a solution if $m_1=m_2$.

If $\varrho q^2 = c_{11} p^2$ (*cf.* (3.23)), then the equations (3.2) may be written

$$\begin{aligned} D\{\lambda_2^2 b\, D f_1 + (c_{12} + \lambda_1^2 b)\, \iota p f_2\} &= 0, \\ \{(c_{11} - \lambda_1^2 b) p^2 + c_{22} D^2\} f_2 + (c_{21} + \lambda_2^2 b)\, \iota p\, D f_1 &= 0. \end{aligned} \tag{4.8}$$

Solving these equations for f_1 and f_2 and using the fact that the displacements must tend to zero as x_2 tends to infinity, we find

$$\begin{aligned} u_1 &= (c_{12} + \lambda_1^2 b)\, \iota p\, (\lambda_2^2 b\, m_1)^{-1} A\, e^{-m_1 x_2} \exp \iota (p x_1 + q t), \\ u_2 &= A\, e^{-m_1 x_2} \exp \iota (p x_1 + q t), \end{aligned} \tag{4.9}$$

where A is a constant and m_1 is the positive root of the equation

$$\lambda_2^2 b\, c_{22} m^2 + \{(c_{11} - \lambda_1^2 b)\, \lambda_2^2 b + (c_{12} + \lambda_1^2 b)(c_{21} + \lambda_2^2 b)\} p^2 = 0. \tag{4.10}$$

In order that the boundary condition (3.12) be satisfied we must have

$$p\,(c_{12} + \lambda_1^2 b - \lambda_2^2 b)\, A = 0. \tag{4.11}$$

Thus, either

$$A = 0, \tag{4.12}$$

or

$$p = 0, \tag{4.13}$$

or

$$c_{12} + \lambda_1^2 b = \lambda_2^2 b. \tag{4.14}$$

If (4.12) is valid, then, from (4.9), $u_1 = u_2 = 0$. If (4.13) is valid, then, from the relation $\varrho q^2 = c_{11} p^2$, we have $q = 0$. Hence, from (4.9) $u_1 = 0$, $u_2 = A\, e^{-m_1 x_2}$. The boundary condition (3.13) then yields

$$c_{22} m_1 A = 0, \tag{4.15}$$

and equation (4.10) reduces to

$$\lambda_2^2 b\, c_{22} m^2 = 0. \tag{4.16}$$

Both (4.15) and (4.16) are satisfied if $m = 0$, in which case u_2 is a constant, or if $c_{22} = 0$ when we have

$$u_1 = 0, \qquad u_2 = A\, e^{-m x_2}, \tag{4.17}$$

where A is arbitrary and m is positive. This is a trivial static solution.

If (4.14) is valid we find $c_{21} = 0$, from (2.6). To satisfy the boundary condition (3.13) we again arrive at (4.15). Then, from (4.10), we find either $p = 0$, or $(c_{11} - \lambda_1^2 b + \lambda_2^2 b)\, \lambda_2^2 b = 0$. The case $p = 0$ has already been discussed. We now have to consider the two subcases (i) $c_{11} = (\lambda_1^2 - \lambda_2^2)\, b$ and (ii) $\lambda_2^2 b = 0$. In subcase (i) we have $c_{11} = -c_{12} = (\lambda_1^2 - \lambda_2^2)\, b$. Then (4.9) gives a solution, provided $\lambda_2^2 b \neq 0$. In subcase (ii) we have $u_1 = 0$, $u_2 = A\, e^{-m_1 x_2} \exp \iota (p x_1 + q t)$. The boundary condition (3.13) is not satisfied unless $m_1 A = 0$. This leads to $u_1 = u_2 = 0$.

Thus, we conclude that if $\varrho q^2 = c_{11} p^2$ and also if $c_{11} = -c_{12} = (\lambda_1^2 - \lambda_2^2)\, b$ the solution given by (4.9) is valid provided $\lambda_2^2 b \neq 0$. In this case A is arbitrary and m_1 is an arbitrary positive number. We note that (4.9) gives a solution with the property that there is no finite depth at which the components u_1 and u_2 of the displacement vanish.

Finally, we discuss the case when $c_{12}+\lambda_1^2 b=0$. From (2.6) and the definition of b, it follows that

$$c_{12}+\lambda_1^2 b = c_{21}+\lambda_2^2 b. \tag{4.18}$$

Hence, in this case, the equations of motion (3.2) reduce to

$$\{(\varrho q^2 - c_{11} p^2) + \lambda_2^2 b D^2\} f_1 = 0, \quad \{(\varrho q^2 - \lambda_1^2 b p^2) + c_{22} D^2\} f_2 = 0. \tag{4.19}$$

The solutions of (4.19) are

$$f_1 = A e^{-m_1 x_2} + A_1 e^{m_1 x_2}, \quad f_2 = B e^{-m_2 x_2} + B_1 e^{m_2 x_2}, \tag{4.20}$$

where m_1 is the positive root of

$$\lambda_2^2 b m_1^2 = c_{11} p^2 - \varrho q^2, \tag{4.21}$$

and m_2 is the positive root of

$$c_{22} m_2^2 = \lambda_1^2 b p^2 - \varrho q^2. \tag{4.22}$$

A, B, A_1 and B_1 are constants. Since the displacement components must tend to zero as x_2 tends to infinity, we obtain from (3.1) and (4.20),

$$u_1 = A e^{-m_1 x_2} \exp \iota(p x_1 + q t), \quad u_2 = B e^{-m_2 x_2} \exp \iota(p x_1 + q t). \tag{4.23}$$

The boundary conditions (3.12) and (3.13) are satisfied, provided

$$m_1 A - \iota p B = 0, \quad \iota p c_{21} A - c_{22} m_2 B = 0. \tag{4.24}$$

Hence, eliminating A and B, we obtain

$$c_{22} m_1 m_2 = - c_{21} p^2. \tag{4.25}$$

Using (3.25), (4.21) and (4.22), we obtain from (4.25),

$$c_{22}^{\frac{1}{2}} (c_{11} \alpha - 1)^{\frac{1}{2}} (\lambda_1^2 b \alpha - 1)^{\frac{1}{2}} = - c_{21} (\lambda_2^2 b)^{\frac{1}{2}} \alpha. \tag{4.26}$$

We note that $\lambda_2^2 b/\varrho$ is the square of a velocity with which a plane wave may be propagated along the x_2-axis in an infinite solid of the material when it is subjected to the same deformation (2.1). Hence, since $c_{21}+\lambda_2^2 b=0$, we have $c_{21}<0$. Also, c_{11}/ϱ and $\lambda_1^2 b/\varrho$ are each the square of a velocity of propagation of a plane wave along the x_1-direction in an infinite body of the material when it is subjected to the deformation (2.1). Accordingly, we conclude from (4.26) that if Rayleigh-type surface waves can be transmitted in the x_1-direction when $c_{12}+\lambda_1^2 b=0$, then the velocity of propagation of these waves must be less than two of the velocities of propagation of plane waves along the x_1-direction when the body is infinite. This result is also true when $c_{12}+\lambda_1^2 b \neq 0$. This follows from the fact that the equation (3.6) must give positive values for both m_1^2 and m_2^2, for otherwise the displacement components (3.7) would not tend to zero as x_2 tends to infinity. Since c_{22} and $\lambda_2^2 b$ are positive, we conclude from (3.6) that necessary conditions for m_1^2 and m_2^2 to be positive are $\varrho q^2 < c_{11} p^2$ and $\varrho q^2 < \lambda_1^2 b p^2$.

5. Second-Order Theory

In this section we derive the changes in the velocities of propagation of Rayleigh surface waves, which are produced by an initial small pure homogeneous deformation of the body, by using the second-order elasticity theory. This is done in the following manner.

The form

$$W = a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3 \tag{5.1}$$

is assumed for the strain-energy function, where

$$J_1 = I_1 - 3, \quad J_2 = I_2 - 2 I_1 + 3, \quad J_3 = I_3 - I_2 + I_1 - 1. \tag{5.2}$$

Then, evaluating W_α, $W_{\alpha\beta}$ and taking

$$\lambda_i = 1 + e_i, \tag{5.3}$$

we use (2.6) and the expression for b following (3.2), to obtain expressions for $c_{11}, c_{22}, c_{12}, c_{21}, \lambda_1^2 b, \lambda_2^2 b$ in which we neglect terms of order higher than the first in the e_i's. The coefficients in (3.26) may then be found. Inserting them, we find the first approximation to the classical frequency equation.

From (5.1) we obtain

$$\begin{aligned}
\frac{\partial W}{\partial I_1} &= \frac{\partial W}{\partial J_1} - 2\frac{\partial W}{\partial J_2} + \frac{\partial W}{\partial J_3} = a_5 - 2a_1 + 2(a_2 - a_3) J_1 + a_3 J_2 + 3 a_4 J_1^2, \\
\frac{\partial W}{\partial I_2} &= \frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3} - a_1 - a_5 + a_3 J_1, \\
\frac{\partial W}{\partial I_3} &= \frac{\partial W}{\partial J_3} = a_5, \\
\frac{\partial^2 W}{\partial I_1^2} &= 2(a_2 - 2a_3) + 6 a_4 J_1, \\
\frac{\partial^2 W}{\partial I_1 \partial I_2} &= a_3, \quad \frac{\partial^2 W}{\partial I_1 \partial I_3} = \frac{\partial^2 W}{\partial I_2 \partial I_3} = \frac{\partial^2 W}{\partial I_2^2} = \frac{\partial^2 W}{\partial I_3^2} = 0.
\end{aligned} \tag{5.4}$$

Using (2.2) and (5.3) we find, on neglecting terms of order higher than the first in the e_i's, that

$$I_1 = 3 + 2(e_1 + e_2 + e_3), \quad I_2 = 3 + 4(e_1 + e_2 + e_3), \quad I_3 = 1 + 2(e_1 + e_2 + e_3), \tag{5.5}$$

and hence, from (5.2), that

$$J_1 = 2(e_1 + e_2 + e_3), \quad J_2 = 0, \quad J_3 = 0. \tag{5.6}$$

Inserting these values of J_α in (5.4) we obtain

$$\begin{aligned}
&W_1 = a_5 - 2a_1 + 4(a_2 - a_3)(e_1 + e_2 + e_3), \\
&W_2 = a_1 - a_5 + 2a_3(e_1 + e_2 + e_3), \quad W_3 = a_5, \\
&W_{11} = 2(a_2 - 2a_3) + 12 a_4 (e_1 + e_2 + e_3), \quad W_{12} = a_3, \\
&W_{13} = W_{23} = W_{22} = W_{33} = 0.
\end{aligned} \tag{5.7}$$

Bearing in mind the last relations of (5.7), $c_{11}, c_{22}, c_{12}, c_{21}$, defined by (2.6), are given by

$$\begin{aligned}
c_{11} &= 2\lambda_1^2 I_3^{-\frac{1}{2}}\{W_1 + (I_1 - \lambda_1^2)\,W_2 + \lambda_2^2\lambda_3^2 W_3 + 2\lambda_1^2 W_{11} + 4\lambda_1^2(I_1 - \lambda_1^2)\,W_{21}\},\\
c_{22} &= 2\lambda_2^2 I_3^{-\frac{1}{2}}\{W_1 + (I_1 - \lambda_2^2)\,W_2 + \lambda_1^2\lambda_3^2 W_3 + 2\lambda_2^2 W_{11} + 4\lambda_2^2(I_1 - \lambda_2^2)\,W_{21}\},\\
c_{12} &= 2\lambda_1^2 I_3^{-\frac{1}{2}}\{-W_1 + (\lambda_2^2 - \lambda_3^2)\,W_2 + \lambda_2^2\lambda_3^2 W_3 + 2\lambda_2^2 W_{11} + 2\lambda_2^2(I_1 + \lambda_3^2)\,W_{21}\},\\
c_{21} &= 2\lambda_2^2 I_3^{-\frac{1}{2}}\{-W_1 + (\lambda_1^2 - \lambda_3^2)\,W_2 + \lambda_1^2\lambda_3^2 W_3 + 2\lambda_1^2 W_{11} + 2\lambda_1^2(I_1 + \lambda_3^2)\,W_{21}\}.
\end{aligned} \tag{5.8}$$

Expanding the terms of (5.8), using (5.3) and (5.7), and neglecting terms of order higher than the first in the e_i's, we find

$$\begin{aligned}
c_{11} &= 4\{2a_2 + 4(2a_2 + 3a_4)\,e_1 + (a_1 + 4a_3 + 12a_4)\,(e_2 + e_3)\},\\
c_{22} &= 4\{2a_2 + (a_1 + 4a_3 + 12a_4)\,(e_1 + e_3) + 4(2a_2 + 3a_4)\,e_2\},\\
c_{12} &= 4\{a_1 + 2a_2 + (a_1 + 4a_3 + 12a_4)\,e_1 + 4(a_3 + 3a_4)\,e_2 +\\
&\quad + 2(-a_1 - 2a_2 + 3a_3 + 6a_4 + a_5)\,e_3\},\\
c_{21} &= 4\{a_1 + 2a_2 + 4(a_3 + 3a_4)\,e_1 + (a_1 + 4a_3 + 12a_4)\,e_2 +\\
&\quad + 2(-a_1 - 2a_2 + 3a_3 + 6a_4 + a_5)\,e_3\}.
\end{aligned} \tag{5.9}$$

From the definition of b, *viz*

$$b = 2I_3^{-\frac{1}{2}}\{W_1 + (I_1 - \lambda_1^2 - \lambda_2^2)\,W_2\},$$

and (5.7), we find

$$\begin{aligned}
\lambda_1^2 b &= 2\{-a_1 + (-a_1 + 4a_2 - 2a_3)\,e_1 + (a_1 + 4a_2 - 2a_3)\,e_2 +\\
&\quad + (3a_1 + 4a_2 - 2a_3 - 2a_5)\,e_3\},\\
\lambda_2^2 b &= 2\{-a_1 + (a_1 + 4a_2 - 2a_3)\,e_1 + (-a_1 + 4a_2 - 2a_3)\,e_2 +\\
&\quad + (3a_1 + 4a_2 - 2a_3 - 2a_5)\,e_3\}.
\end{aligned} \tag{5.10}$$

We now evaluate the coefficients in the frequency equation (3.26), *viz*

$$\begin{aligned}
&(1 - c_{11}\alpha)\{1 - (\lambda_1^2 - \lambda_2^2)\,b\,\alpha\}^2\,\lambda_2^2\,b\,c_{22}\\
&\qquad = (1 - \lambda_1^2 b\,\alpha)\{[c_{11}c_{22} - c_{12}c_{21} - c_{21}(\lambda_1^2 - \lambda_2^2)\,b]\,\alpha - c_{22}\}^2.
\end{aligned} \tag{5.11}$$

We note from (5.10) that

$$(\lambda_1^2 - \lambda_2^2)\,b = -4a_1(e_1 - e_2). \tag{5.12}$$

We may therefore neglect terms involving $(\lambda_1^2 - \lambda_2^2)^2 b^2$ when expanding (5.11). We find

$$\begin{aligned}
&\lambda_1^2 b\{c_{11}c_{22} - c_{12}c_{21} - c_{21}(\lambda_1^2 - \lambda_2^2)\,b\}^2\alpha^3 - \{[c_{11}c_{22} - c_{12}c_{21} - c_{21}(\lambda_1^2 - \lambda_2^2)\,b]^2 +\\
&\quad + 2\lambda_1^2 b\,c_{22}[c_{11}c_{22} - c_{12}c_{21} - c_{21}(\lambda_1^2 - \lambda_2^2)\,b] - 2\lambda_2^2 b\,c_{11}c_{22}(\lambda_1^2 - \lambda_2^2)\,b\}\,\alpha^2 +\\
&\quad + \{2c_{22}[c_{11}c_{22} - c_{12}c_{21} - c_{21}(\lambda_1^2 - \lambda_2^2)\,b] + \lambda_1^2 b\,c_{22}^2 -\\
&\quad - \lambda_2^2 b\,c_{22}[c_{11} + 2(\lambda_1^2 - \lambda_2^2)\,b]\}\,\alpha - c_{22}^2 + c_{22}\lambda_2^2 b = 0.
\end{aligned} \tag{5.13}$$

Using (5.9), (5.10) and (5.12), we find the coefficients in (5.13). That of α^3 is

$$\begin{aligned} & 512 a_1^2 (a_1 + 4a_2) \{- a_1 (a_1 + 4a_2) + \\ & \quad + (- a_1^2 + 48 a_2^2 + 4 a_1 a_2 - 18 a_1 a_3 - 48 a_1 a_4 - 24 a_2 a_3)\, e_1 + \\ & \quad + (- 3 a_1^2 + 48 a_2^2 + 4 a_1 a_2 - 18 a_1 a_3 - 48 a_1 a_4 - 24 a_2 a_3)\, e_2 + \\ & \quad + (11 a_1^2 + 48 a_2^2 + 56 a_1 a_2 - 26 a_1 a_3 - 48 a_1 a_4 - \\ & \quad - 10 a_1 a_5 - 24 a_2 a_3 - 24 a_2 a_5)\, e_3\}; \end{aligned} \tag{5.14}$$

that of α^2 is

$$\begin{aligned} & - 256 \{a_1^2 (a_1 + 4a_2)(a_1 + 6a_2) - \\ & - [4 a_1 a_2 (48 a_2^2 + 14 a_1 a_2 - 29 a_1 a_3 - 72 a_1 a_4 - 24 a_2 a_3) + \\ & \qquad + a_1^2 (- a_1^2 - 2 a_1 a_2 - 20 a_1 a_3 - 60 a_1 a_4)]\, e_1 - \\ & - [4 a_1 a_2 (48 a_2^2 - 2 a_1 a_2 - 25 a_1 a_3 - 72 a_1 a_4 - 24 a_2 a_3) + \\ & \qquad + 2 a_1^2 (- 2 a_1^2 - 15 a_1 a_2 - 8 a_1 a_3 - 30 a_1 a_4)]\, e_2 - \\ & - [4 a_1 a_2 (48 a_2^2 + 66 a_1 a_2 - 39 a_1 a_3 - 72 a_1 a_4 - 15 a_1 a_5 - 24 a_2 a_3 - 24 a_2 a_5) + \\ & \qquad + a_1^2 (7 a_1^2 + 82 a_1 a_2 - 28 a_1 a_3 - 60 a_1 a_4 - 8 a_1 a_5)]\, e_3\}; \end{aligned} \tag{5.15}$$

that of α is

$$\begin{aligned} & 64 \{- 4 a_1 a_2 (a_1 + 4a_2) - \\ & \quad - (2 a_1^3 - 64 a_2^3 + 11 a_1^2 a_2 + 8 a_1^2 a_3 + 24 a_1^2 a_4 - 12 a_1 a_2^2 + 32 a_2^2 a_3 + \\ & \qquad + 68 a_1 a_2 a_3 + 192 a_1 a_2 a_4)\, e_1 - \\ & \quad - (- 64 a_2^3 + 21 a_1^2 a_2 + 24 a_1^2 a_4 + 76 a_1 a_2^2 + 32 a_2^2 a_3 + \\ & \qquad + 28 a_1 a_2 a_3 + 192 a_1 a_2 a_4)\, e_2 \\ & \quad - (2 a_1^3 - 64 a_2^3 - 8 a_1^2 a_2 + 8 a_1^2 a_3 + 24 a_1^2 a_4 - 80 a_1 a_2^2 + 32 a_2^2 a_3 + \\ & \qquad + 32 a_2^2 a_5 + 80 a_1 a_2 a_3 + 192 a_1 a_2 a_4 + 16 a_1 a_2 a_5)\, e_3\}, \end{aligned} \tag{5.16}$$

and that of α^0 is

$$\begin{aligned} & - 8 \{2 a_2 (a_1 + 4a_2) + \\ & + (a_1^2 - 8 a_2^2 + 6 a_1 a_2 + 4 a_1 a_3 + 12 a_1 a_4 + 36 a_2 a_3 + 96 a_2 a_4)\, e_1 + \\ & + (56 a_2^2 + 10 a_1 a_2 + 12 a_1 a_4 + 4 a_2 a_3 + 96 a_2 a_4)\, e_2 + \\ & + (a_1^2 - 8 a_2^2 + 2 a_1 a_2 + 4 a_1 a_3 + 12 a_1 a_4 + 36 a_2 a_3 + 96 a_2 a_4 + 4 a_2 a_5)\, e_3\}. \end{aligned} \tag{5.17}$$

In solving (5.13) we must bear in mind equation (3.9), *viz*

$$W_1 + (I_1 - \lambda_2^2) W_2 + \lambda_1^2 \lambda_3^2 W_3 = 0, \tag{5.18}$$

which, with the aid of (5.7), yields

$$e_3 = - e_1 - \{2 a_2 / (a_1 + 2 a_2)\}\, e_2. \tag{5.19}$$

We eliminate e_3 from the expressions (5.14), (5.15), (5.16), (5.17). The resulting expressions are linear in e_1 and e_2. We denote the coefficient of α^3 by $\bar{a} + \bar{a}_1 e_1 + \bar{a}_2 e_2$, that of α^2 by $\bar{b} + \bar{b}_1 e_1 + \bar{b}_2 e_2$, that of α by $\bar{c} + \bar{c}_1 e_1 + \bar{c}_2 e_2$ and the constant term is denoted by $\bar{d}+\bar{d}_1 e_1+\bar{d}_2 e_2$. Then equation (5.13) may be written

$$(\bar{a}+\bar{a}_1 e_1+\bar{a}_2 e_2)\,\alpha^3+(\bar{b}+\bar{b}_1 e_1+\bar{b}_2 e_2)\,\alpha^2+(\bar{c}+\bar{c}_1 e_1+\bar{c}_2 e_2)\,\alpha+ \\ +\bar{d}+\bar{d}_1 e_1+\bar{d}_2 e_2=0, \tag{5.20}$$

where, using (5.14), (5.15), (5.16) and (5.17),

$$\begin{aligned}
\bar{a} &= -512 a_1^3 (a_1+4a_2)^2,\\
\bar{b} &= -256 a_1^2 (a_1+4a_2)(a_1+6a_2),\\
\bar{c} &= -256 a_1 a_2 (a_1+4a_2),\\
\bar{d} &= -16 a_2 (a_1+4a_2),\\
\bar{a}_1 &= 512 a_1^2 (a_1+4a_2)(-12a_1^2-52a_1 a_2+8a_1 a_3+10a_1 a_5+24a_2 a_5),\\
\bar{b}_1 &= 256\{4a_1 a_2(-10a_1^2-52a_1 a_2+10a_1 a_3+10a_1 a_5+24a_2 a_5)+\\
&\quad +a_1^2(-8a_1^2-44a_1 a_2+8a_1 a_3+8a_1 a_5+20a_2 a_5)\},\\
\bar{c}_1 &= -64 a_2 (19a_1^2+68a_1 a_2-32a_2 a_5-12a_1 a_3-16a_1 a_5),\\
\bar{d}_1 &= -32(a_1 a_2-a_2 a_5),\\
\bar{a}_2 &= 512 a_1^2 (a_1+4a_2)(-3a_1^3-24a_1^2 a_2-18a_1^2 a_3-48a_1^2 a_4-56a_1 a_2^2+\\
&\quad +48a_2^2 a_5-8a_1 a_2 a_3+20a_1 a_2 a_5)(a_1+2a_2)^{-1},\\
\bar{b}_2 &= -512 a_1 (2a_1^4+26a_1^3 a_2+8a_1^3 a_3+30a_1^3 a_4+176a_1 a_2^3-96a_2^3 a_5+\\
&\quad +116a_1^2 a_2^2+38a_1^2 a_2 a_3+144a_1^2 a_2 a_4-8a_1 a_2^2 a_3-\\
&\quad -8a_1^2 a_2 a_5-60a_1 a_2^2 a_5)(a_1+2a_2)^{-1},\\
\bar{c}_2 &= -64(17a_1^3 a_2+24a_1^3 a_4+248a_1 a_2^3-64a_2^3 a_5+134a_1^2 a_2^2+12a_1^2 a_2 a_3+\\
&\quad +192a_1^2 a_2 a_4-72a_1 a_2^2 a_3-32a_1 a_2^2 a_5)(a_1+2a_2)^{-1},\\
\bar{d}_2 &= -8(128a_2^3+8a_1^2 a_2+12a_1^2 a_4+72a_1 a_2^2-64a_2^2 a_3-8a_2^2 a_5-\\
&\quad -4a_1 a_2 a_3+96a_1 a_2 a_4)(a_1+2a_2)^{-1}.
\end{aligned} \tag{5.21}$$

We write

$$\alpha=\alpha_0+\alpha_1 e_1+\alpha_2 e_2, \tag{5.22}$$

where $\alpha_0, \alpha_1, \alpha_2$ are independent of e_1, e_2 and are to be determined. Inserting (5.22) in (5.20) and expanding, we find, on neglecting terms of order higher than the first in e_1, e_2, that

$$\begin{aligned}
&\bar{a}\,\alpha_0^3+\bar{b}\,\alpha_0^2+\bar{c}\,\alpha_0+\bar{d}+\\
&\quad +e_1\{\alpha_1(3\,\bar{a}\,\alpha_0^2+2\,\bar{b}\,\alpha_0+\bar{c})+\bar{a}_1\alpha_0^3+\bar{b}_1\alpha_0^2+\bar{c}_1\alpha_0+\bar{d}_1\}+\\
&\quad +e_2\{\alpha_2(3\,\bar{a}\,\alpha_0^2+2\,\bar{b}\,\alpha_0+\bar{c})+\bar{a}_2\alpha_0^3+\bar{b}_2\alpha_0^2+\bar{c}_2\alpha_0+\bar{d}_2\}=0.
\end{aligned} \tag{5.23}$$

Hence, since e_1, e_2 are independent, we have

$$\bar{a}\,\alpha_0^3+\bar{b}\,\alpha_0^2+\bar{c}\,\alpha_0+\bar{d}=0; \tag{5.24}$$

$$\begin{aligned}
\alpha_1(3\,\bar{a}\,\alpha_0^2+2\,\bar{b}\,\alpha_0+\bar{c})+\bar{a}_1\alpha_0^3+\bar{b}_1\alpha_0^2+\bar{c}_1\alpha_0+\bar{d}_1=0,\\
\alpha_2(3\,\bar{a}\,\alpha_0^2+2\,\bar{b}\,\alpha_0+\bar{c})+\bar{a}_2\alpha_0^3+\bar{b}_2\alpha_0^2+\bar{c}_2\alpha_0+\bar{d}_2=0.
\end{aligned} \tag{5.25}$$

Equation (5.24) is the equation found in classical elasticity theory for the frequency of Rayleigh surface waves. This may be seen if we denote $4(a_1+2a_2)$ by $\bar{\lambda}$, $-2a_1$ by $\bar{\mu}$, where $\bar{\lambda}$, $\bar{\mu}$ are the Lamé constants. Then, from (5.21), we have

$$\begin{aligned} \bar{a} &= 16\bar{\mu}^3(\bar{\lambda}+\bar{\mu})^2, \\ \bar{b} &= -8\bar{\mu}^2(\bar{\lambda}+\bar{\mu})(3\bar{\lambda}+4\bar{\mu}), \\ \bar{c} &= 8\bar{\mu}(\bar{\lambda}+\bar{\mu})(\bar{\lambda}+2\bar{\mu}), \\ \bar{d} &= -(\bar{\lambda}+\bar{\mu})(\bar{\lambda}+2\bar{\mu}). \end{aligned} \tag{5.26}$$

Inserting these values in (5.24) and dividing throughout by $(\bar{\lambda}+\bar{\mu})$, we find

$$16\bar{\mu}^3(\bar{\lambda}+\bar{\mu})\,\alpha_0^3 - 8\bar{\mu}^2(3\bar{\lambda}+4\bar{\mu})\,\alpha_0^2 + 8\bar{\mu}(\bar{\lambda}+2\bar{\mu})\,\alpha_0 - (\bar{\lambda}+2\bar{\mu}) = 0. \tag{5.27}$$

Taking $\bar{\mu}\,\alpha_0 = \bar{\alpha}_0$, this equation may be written

$$8\bar{\alpha}_0^3 - 8(2-\sigma)\,\bar{\alpha}_0^2 + 8(1-\sigma)\,\bar{\alpha}_0 - (1-\sigma) = 0, \tag{5.28}$$

where σ, Poisson's ratio, is given by

$$\sigma = \bar{\lambda}/2(\bar{\lambda}+\bar{\mu}).$$

(5.28) is the equation found in the classical theory [2].

6. The Equation for *m*

In this section we derive the first approximation to the equation (3.6) for m. From (3.6) and (4.18) we obtain

$$\begin{aligned} \lambda_2^2 b\, c_{22}\, m^4 + \{(c_{22}+\lambda_2^2 b)\,\varrho q^2 - [c_{11}c_{22} + \lambda_1^2\lambda_2^2 b^2 - (c_{12}+\lambda_1^2 b)^2]\,p^2\}\, m^2 + \\ + (\varrho q^2 - c_{11}p^2)(\varrho q^2 - \lambda_1^2 b\, p^2) = 0. \end{aligned} \tag{6.1}$$

Writing

$$p^2/\varrho q^2 = \alpha, \qquad m^2/\varrho q^2 = \beta^2, \tag{6.2}$$

(6.1) may be written

$$\begin{aligned} \lambda_2^2 b\, c_{22}\, \beta^4 + \{c_{22} + \lambda_2^2 b - [c_{11}c_{22} + \lambda_1^2\lambda_2^2 b^2 - (c_{12}+\lambda_1^2 b)^2]\,\alpha\}\,\beta^2 + \\ + (1 - c_{11}\alpha)(1 - \lambda_1^2 b\,\alpha) = 0. \end{aligned} \tag{6.3}$$

We write

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2, \\ \beta &= \beta_0 + \beta_1 e_1 + \beta_2 e_2, \\ \lambda_2^2 b\, c_{22} &= R_0 + R_1 e_1 + R_2 e_2, \\ c_{22} + \lambda_2^2 b &= S_0 + S_1 e_1 + S_2 e_2, \\ c_{11}c_{22} + \lambda_1^2\lambda_2^2 b^2 - (c_{12}+\lambda_1^2 b)^2 &= \tau_0 + \tau_1 e_1 + \tau_2 e_2, \\ \lambda_1^2 b\, c_{11} &= \varkappa_0 + \varkappa_1 e_1 + \varkappa_2 e_2, \\ c_{11} + \lambda_1^2 b &= \omega_0 + \omega_1 e_1 + \omega_2 e_2, \end{aligned} \tag{6.4}$$

where $\alpha_0, \alpha_1, \alpha_2$ are determined by equations (5.24) and (5.25); $\beta_0, \beta_1, \beta_2$ are independent of e_1, e_2 and are to be determined; $R, S, \ldots, \omega$, $R_1, S_1, \ldots, \omega_1$,

$R_2, S_2, \ldots, \omega_2$ are given by

$$\begin{aligned}
R_0 &= -16a_1a_2, \\
S_0 &= -2a_1 + 8a_2, \\
\tau_0 &= -32a_1a_2, \\
\varkappa_0 &= -16a_1a_2, \\
\omega_0 &= -2a_1 + 8a_2, \\
R_1 &= -32a_2(a_1 - a_5), \\
S_1 &= 4(-a_1 + a_5), \\
\tau_1 &= 8(a_1^2 - 20a_1a_2 + 4a_1a_3 + 8a_2a_5), \\
\varkappa_1 &= 8(a_1^2 - 16a_1a_2 + 4a_1a_3 + 4a_2a_5), \\
\omega_1 &= 2(-6a_1 + 16a_2 - 8a_3 + 2a_5), \\
R_2 &= 32(-2a_1^2a_2 - 3a_1^2a_4 - 6a_1a_2^2 + a_1a_2a_3 + 2a_2^2a_5)(a_1 + 2a_2)^{-1}, \\
S_2 &= 2(-a_1^2 + 32a_2^2 + 8a_1a_2 - 2a_1a_3 + 24a_1a_4 - 16a_2a_3 + 4a_2a_5)(a_1 + 2a_2)^{-1}, \\
\tau_2 &= 8(-a_1^3 - 6a_1^2a_2 - 4a_1^2a_3 - 24a_1^2a_4 - 24a_1a_2^2 + 16a_2^2a_5)(a_1 + 2a_2)^{-1}, \\
\varkappa_2 &= 8(-a_1^3 + 2a_1^2a_2 - 4a_1^2a_3 - 12a_1^2a_4 + 8a_2^2a_5 - 4a_1a_2a_3)(a_1 + 2a_2)^{-1}, \\
\omega_2 &= 2(3a_1^2 + 6a_1a_3 + 24a_1a_4 + 4a_2a_5)(a_1 + 2a_2)^{-1}.
\end{aligned} \tag{6.5}$$

In deriving (6.5) we have used (5.9), (5.10) and (5.19). Inserting the values (6.4) in (6.3) and expanding, with the neglect of terms of order higher than the first in e_1, e_2, we find

$$\begin{aligned}
&R_0\beta_0^4 + (S_0 - \tau_0\alpha_0)\beta_0^2 + 1 - \omega_0\alpha_0 + \varkappa_0\alpha_0^2 + \\
&\quad + e_1\{(4R_0\beta_0^3 + 2S_0\beta_0 - 2\tau_0\alpha_0\beta_0)\beta_1 + R_1\beta_0^4 + (S_1 - \tau_1\alpha_0 - \tau_0\alpha_1)\beta_0^2 + \\
&\quad + \varkappa_1\alpha_0^2 + 2\varkappa_0\alpha_0\alpha_1 - \omega_0\alpha_1 - \omega_1\alpha_0\} + \\
&\quad + e_2\{(4R_0\beta_0^3 + 2S_0\beta_0 - 2\tau_0\alpha_0\beta_0)\beta_2 + R_2\beta_0^4 + (S_2 - \tau_2\alpha_0 - \tau_0\alpha_2)\beta_0^2 + \\
&\quad + \varkappa_2\alpha_0^2 + 2\varkappa_0\alpha_0\alpha_2 - \omega_0\alpha_2 - \omega_2\alpha_0\} = 0.
\end{aligned} \tag{6.6}$$

Since e_1, e_2 are independent, this equation gives

$$R_0\beta_0^4 + (S_0 - \tau_0\alpha_0)\beta_0^2 + 1 - \omega_0\alpha_0 + \varkappa_0\alpha_0^2 = 0; \tag{6.7}$$

$$\begin{aligned}
&(4R_0\beta_0^3 + 2S_0\beta_0 - 2\tau_0\alpha_0\beta_0)\beta_1 + R_1\beta_0^4 + (S_1 - \tau_1\alpha_0 - \tau_0\alpha_1)\beta_0^2 + \\
&\quad + \varkappa_1\alpha_0^2 + 2\varkappa_0\alpha_0\alpha_1 - \omega_0\alpha_1 - \omega_1\alpha_0 = 0,
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
&(4R_0\beta_0^3 + 2S_0\beta_0 - 2\tau_0\alpha_0\beta_0)\beta_2 + R_2\beta_0^4 + (S_2 - \tau_2\alpha_0 - \tau_0\alpha_2)\beta_0^2 + \\
&\quad + \varkappa_2\alpha_0^2 + 2\varkappa_0\alpha_0\alpha_2 - \omega_0\alpha_2 - \omega_2\alpha_0 = 0.
\end{aligned} \tag{6.9}$$

It is assumed that $\alpha_0, \alpha_1, \alpha_2$ are known from equations (5.24) and (5.25). Having found β_0 from (6.7) we may then solve (6.8) and (6.9) for β_1 and β_2.

Equation (6.7), which is the equation found in the classical theory, may readily be solved. Inserting the values of $R_0, S_0, \ldots, \omega_0$ from (6.5) in (6.7) we

find
$$16 a_1 a_2 \beta_0^4 - (-2a_1 + 8a_2 + 32 a_1 a_2 \alpha_0)\beta_0^2 - \\ - \{1 + (2a_1 - 8a_2)\alpha_0 - 16 a_1 a_2 \alpha_0^2\} = 0. \tag{6.10}$$

Denoting the roots by β_{01}, β_{02} we see that they are given by
$$\beta_{01}^2 = \alpha_0 - \bar{\mu}^{-1}, \qquad \beta_{02}^2 = \alpha_0 - (\bar{\lambda} + 2\bar{\mu})^{-1}, \tag{6.11}$$
where we have written $4a_1 + 8a_2 = \bar{\lambda}$, $-2a_1 = \bar{\mu}$.

7. Results for an Incompressible Body

In this section the frequency equation and the equation for m, found in Sections 5 and 6, for a compressible body on the basis of the second-order theory, are used in deducing the corresponding second-order equations for an incompressible body. The resulting equations are solved in explicit form and expressions for the displacement components are then obtained.

RIVLIN [*4*] has shown that to deduce the second-order theory results for an incompressible body from those for a compressible one, the quantity $(e_1 + e_2 + e_3)$ must tend to zero and a_2, a_3 must tend to infinity in such a way that the product $a_2(e_1 + e_2 + e_3)$ and the difference $a_3 - 2a_2$ remain finite. The equation (5.19), *viz*
$$e_3 = -e_1 - \{2a_2/(a_1 + 2a_2)\} e_2,$$
yields $e_1 + e_2 + e_3 = 0$, if we let a_2 tend to infinity, a_1 being finite. Thus we see that e_1 and e_2 may still be regarded as independent. Hence, the frequency equation for an incompressible body may be deduced from (5.24) and (5.25) by letting a_2, a_3 tend to infinity but keeping $a_3 - 2a_2$ finite. In the same way the equation for m may be found from (6.7), (6.8) and (6.9).

Proceeding in this manner, (5.24) and (5.25) become
$$128 a_1^3 \alpha_0^3 + 96 a_1^2 \alpha_0^2 + 16 a_1 \alpha_0 + 1 = 0; \tag{7.1}$$
$$\begin{aligned} &\alpha_1(-96 a_1^3 \alpha_0^2 - 48 a_1^2 \alpha_0 - 4a_1) + 96 a_1^2(-3a_1 + 2a_5)\alpha_0^3 + \\ &\quad + 32 a_1(-4a_1 + 3a_5)\alpha_0^2 - (11 a_1 - 8a_5)\alpha_0 = 0, \\ &\alpha_2(-96 a_1^3 \alpha_0^2 - 48 a_1^2 \alpha_0 - 4a_1) + 96 a_1^2(-3a_1 + 2a_5)\alpha_0^3 + \\ &\quad + 32 a_1(-5a_1 + 3a_5)\alpha_0^2 - (13 a_1 - 8a_5)\alpha_0 = 0. \end{aligned} \tag{7.2}$$

Equation (7.1) is the frequency equation found in classical elasticity theory for an incompressible body. It has a real root given by $a_1\alpha_0 = -0.54787 \ldots$ The remaining two roots are complex and do not give rise to Rayleigh waves [*2*].

The constants a_1 and a_5 are related to the constants C_1 and C_2 which occur in the expression for the strain-energy function for a Mooney material, *viz*
$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \tag{7.3}$$
by
$$a_1 = -(C_1 + C_2), \qquad a_5 = -(C_1 + 2C_2). \tag{7.4}$$

Thus the real root α_0 of (7.1) may be written
$$\alpha_0 = 0.54787 (C_1 + C_2)^{-1}. \tag{7.5}$$

Using (7.4) and (7.5), we obtain from (7.2)

$$\begin{aligned} \alpha_1 &= -(1.2007\,C_1 + 0.1049\,C_2)\,(C_1+C_2)^{-2}, \\ \alpha_2 &= (0.1049\,C_1 + 1.2007\,C_2)\,(C_1+C_2)^{-2}. \end{aligned} \tag{7.6}$$

For future reference, we obtain from (7.5) and (7.6),

$$\begin{aligned} \alpha_1\alpha_0^{-1} &= -(2.1915\,C_1 + 0.1915\,C_2)\,(C_1+C_2)^{-1}, \\ \alpha_2\alpha_0^{-1} &= (0.1915\,C_1 + 2.1915\,C_2)\,(C_1+C_2)^{-1}. \end{aligned} \tag{7.7}$$

From (6.7), (6.8) and (6.9), we deduce

$$2a_1\beta_0^4 - (1+4a_1\alpha_0)\,\beta_0^2 + \alpha_0(1+2a_1\alpha_0) = 0, \tag{7.8}$$

$$\begin{aligned} (2 - 8a_1\beta_0^2 + 8a_1\alpha_0)\,\beta_0\beta_1 + 4(-a_1+a_5)\,\beta_0^4 + \{4(3a_1-2a_5)\,\alpha_0 + {} \\ + 4a_1\alpha_1\}\,\beta_0^2 + 4(-2a_1+a_5)\,\alpha_0^2 - 4a_1\alpha_0\alpha_1 - \alpha_1 = 0, \end{aligned} \tag{7.9}$$

$$\begin{aligned} (2 - 8a_1\beta_0^2 + 8a_1\alpha_0)\,\beta_0\beta_2 + 4(-2a_1+a_5)\,\beta_0^4 + \{4(3a_1-2a_5)\,\alpha_0 + {} \\ + 4a_1\alpha_2\}\,\beta_0^2 + 4(-a_1+a_5)\,\alpha_0^2 - 4a_1\alpha_0\alpha_2 - \alpha_2 = 0. \end{aligned} \tag{7.10}$$

Equation (7.8) has roots $\pm\beta_{01}$, $\pm\beta_{02}$ given by

$$\beta_{01}^2 = \alpha_0, \qquad \beta_{02}^2 = \alpha_0 + \tfrac{1}{2}\,a_1^{-1}. \tag{7.11}$$

Thus, using (7.4) and (7.5), we obtain

$$\begin{aligned} \beta_{01} &= 0.74018\,(C_1+C_2)^{-\frac{1}{2}}, \\ \beta_{02} &= 0.2188\,(C_1+C_2)^{-\frac{1}{2}}. \end{aligned} \tag{7.12}$$

We denote the values of β_1, β_2 corresponding to $\beta_{01}^2 = \alpha_0$, by β_{11}, β_{21} respectively, and those corresponding to $\beta_{02}^2 = \alpha_0 + \frac{1}{2}a_1^{-1}$, by β_{12}, β_{22} respectively. Inserting $\beta_0^2 = \beta_{01}^2 = \alpha_0$, in (7.9) and (7.10), we find

$$2\beta_{01}\beta_{11} - \alpha_1 = 0, \qquad 2\beta_{01}\beta_{21} - \alpha_2 = 0. \tag{7.13}$$

Likewise, inserting $\beta_0^2 = \beta_{02}^2 = \alpha_0 + \frac{1}{2}a_1^{-1}$, in (7.9) and (7.10), we obtain

$$\begin{aligned} 2a_1^2\beta_{02}\beta_{12} - 2a_1^2\alpha_0 - a_1^2\alpha_1 + a_1 - a_5 &= 0, \\ 2a_1^2\beta_{02}\beta_{22} + 2a_1^2\alpha_0 - a_1^2\alpha_2 + 2a_1 - a_5 &= 0. \end{aligned} \tag{7.14}$$

Using (7.1) and (7.2), it follows from (7.14) that

$$\begin{aligned} \beta_{12} &= \tfrac{1}{2}\,\beta_{02}\alpha_1\alpha_0^{-1}, \\ \beta_{22} &= \tfrac{1}{2}\,\beta_{02}\alpha_2\alpha_0^{-1}. \end{aligned} \tag{7.15}$$

By (7.6), (7.7) and (7.12), we obtain from (7.13)

$$\begin{aligned} \beta_{11} &= -(0.81104\,C_1 + 0.07086\,C_2)\,(C_1+C_2)^{-\frac{3}{2}}, \\ \beta_{21} &= (0.07086\,C_1 + 0.81104\,C_2)\,(C_1+C_2)^{-\frac{3}{2}}, \end{aligned} \tag{7.16}$$

and from (7.15),

$$\begin{aligned} \beta_{12} &= -(0.2397\,C_1 + 0.0209\,C_2)\,(C_1+C_2)^{-\frac{3}{2}}, \\ \beta_{22} &= (0.0209\,C_1 + 0.2397\,C_2)\,(C_1+C_2)^{-\frac{3}{2}}. \end{aligned} \tag{7.17}$$

Finally, we obtain expressions for the displacement components. For convenience in the calculations we replace (3.7) by

$$\begin{aligned} u_1 &= \iota p\,(m_1 A\,e^{-m_1 x_2} + m_2 B\,e^{-m_2 x_2}) \exp \iota (p\,x_1 + q\,t)\,, \\ u_2 &= (c_{12} + \lambda_1^2 b)^{-1} \{\varrho\,q^2 - c_{11} p^2 + m_1^2 \lambda_2^2 b)\,A\,e^{-m_1 x_2} + \\ &\quad + (\varrho\,q^2 - c_{11} p^2 + m_2^2 \lambda_2^2 b)\,B\,e^{-m_2 x_2}\} \exp \iota (p\,x_1 + q\,t)\,. \end{aligned} \tag{7.18}$$

These values of u_1 and u_2 satisfy the equations of motion and give rise to the same frequency equation as do the values given by (3.7). The limiting value of $c_{11}(c_{12} + \lambda_1^2 b)^{-1}$ is 1, and that of $\lambda_2^2 b\,(c_{12} + \lambda_1^2 b)^{-1}$ is 0. Hence the limiting form of (7.18) is

$$\begin{aligned} u_1 &= \iota p\,(m_1 A\,e^{-m_1 x_2} + m_2 B\,e^{-m_2 x_2}) \exp \iota (p\,x_1 + q\,t)\,, \\ u_2 &= -\,\varrho\,q^2 (\alpha_0 + \alpha_1 e_1 + \alpha_2 e_2)\,(A\,e^{-m_1 x_2} + B\,e^{-m_2 x_2}) \exp \iota (p\,x_1 + q\,t)\,. \end{aligned} \tag{7.19}$$

It is to be understood here that the period $2\pi/q$ is assumed to be given and that it is independent of e_1 and e_2, that p, m_1 and m_2 take their limiting values and that A and B depend on e_1 and e_2. Since the body is incompressible, the density ϱ does not change. The dependence of A and B on e_1 and e_2 is given by (3.16), *viz*

$$\begin{aligned} &\{(\varrho\,q^2 - c_{11} p^2) + m_1^2 [(\lambda_2^2 - \lambda_1^2)\,b - c_{12}]\}\,A \\ &\qquad = -\,\{(\varrho\,q^2 - c_{11} p^2) + m_2^2 [(\lambda_2^2 - \lambda_1^2)\,b - c_{12}]\}\,B\,. \end{aligned} \tag{7.20}$$

We write

$$A = A_0 + A_1 e_1 + A_2 e_2\,, \qquad B = B_0 + B_1 e_1 + B_2 e_2\,, \tag{7.21}$$

where A_0, A_1, … B_2 are independent of e_1 and e_2. Using (7.21), (7.20) may be written

$$\begin{aligned} &(A_0 + A_1 e_1 + A_2 e_2)\,\{(1 - \alpha_0 \zeta_0 - \eta_0 \beta_{01}^2) - e_1 (\alpha_0 \zeta_1 + \alpha_1 \zeta_0 + 2\eta_0 \beta_{01} \beta_{11} + \\ &\qquad + \eta_1 \beta_{01}^2) - e_2 (\alpha_0 \zeta_2 + \alpha_2 \zeta_0 + 2\eta_0 \beta_{01} \beta_{21} + \eta_2 \beta_{01}^2)\} \\ &\quad = -\,(B_0 + B_1 e_1 + B_2 e_2)\,\{(1 - \alpha_0 \zeta_0 - \eta_0 \beta_{02}^2) - e_1 (\alpha_0 \zeta_1 + \alpha_1 \zeta_0 + \\ &\qquad + 2\eta_0 \beta_{02} \beta_{12} + \eta_1 \beta_{02}^2) - e_2 (\alpha_0 \zeta_2 + \alpha_2 \zeta_0 + 2\eta_0 \beta_{02} \beta_{22} + \eta_2 \beta_{02}^2)\}\,, \end{aligned} \tag{7.22}$$

where we have written

$$c_{11} = \zeta_0 + \zeta_1 e_1 + \zeta_2 e_2\,, \qquad (\lambda_2^2 - \lambda_1^2)\,b - c_{12} = -\,\eta_0 - \eta_1 e_1 - \eta_2 e_2\,, \tag{7.23}$$

$\zeta_0, \zeta_1, \ldots, \eta_2$ being given by

$$\begin{aligned} &\zeta_0 = 8 a_2\,, \qquad \zeta_1 = 4\,(-\,a_1 + 8 a_2 - 4 a_3)\,, \\ &\zeta_2 = 4 a_1 (a_1 + 4 a_3 + 12 a_4)\,(a_1 + 2 a_2)^{-1}, \\ &\eta_0 = 4\,(a_1 + 2 a_2)\,, \qquad \eta_1 = 8\,(a_1 + 2 a_2 - a_3 - a_5)\,, \\ &\eta_2 = 4\,(a_1^2 + 8 a_2^2 + 6 a_1 a_2 + 4 a_1 a_3 + 12 a_1 a_4 - 4 a_2 a_3 - 4 a_2 a_5)\,(a_1 + 2 a_2)^{-1}. \end{aligned} \tag{7.24}$$

In obtaining (7.24), we have used (5.9), (5.10) and (5.19). We observe that if we let a_2, a_3 tend to infinity while keeping $a_3 - 2a_2$ finite, then $a_2^{-1} \zeta_0$ and $a_2^{-1} \eta_0$ tend to 8, and $a_2^{-1} \zeta_1$, $a_2^{-1} \zeta_2$, $a_2^{-1} \eta_1$ and $a_2^{-1} \eta_2$ tend to zero. Hence the limiting form of (7.22) is

$$\begin{aligned} &(A_0 + A_1 e_1 + A_2 e_2)\,\{\alpha_0 + \beta_{01}^2 + e_1 (\alpha_1 + 2\beta_{01} \beta_{11}) + e_2 (\alpha_2 + 2\beta_{01} \beta_{21})\} \\ &= -\,(B_0 + B_1 e_1 + B_2 e_2)\,\{\alpha_0 + \beta_{02}^2 + e_1 (\alpha_1 + 2\beta_{02} \beta_{12}) + e_2 (\alpha_2 + 2\beta_{02} \beta_{22})\}\,. \end{aligned} \tag{7.25}$$

Thus, since e_1 and e_2 are independent, we have

$$(\alpha_0 + \beta_{01}^2) A_0 = -(\alpha_0 + \beta_{02}^2) B_0; \tag{7.26}$$

$$\begin{aligned}(\alpha_0 + \beta_{01}^2) A_1 + (\alpha_1 + 2\beta_{01}\beta_{11}) A_0 &= -(\alpha_0 + \beta_{02}^2) B_1 - (\alpha_1 + 2\beta_{02}\beta_{12}) B_0,\\ (\alpha_0 + \beta_{01}^2) A_2 + (\alpha_2 + 2\beta_{01}\beta_{21}) A_0 &= -(\alpha_0 + \beta_{02}^2) B_2 - (\alpha_2 + 2\beta_{02}\beta_{22}) B_0.\end{aligned} \tag{7.27}$$

Using (7.11), (7.26) may be written

$$A_0 = -\tfrac{1}{2}\alpha_0^{-1}(\alpha_0 + \beta_{02}^2) B_0 = -(1 + \tfrac{1}{4} a_1^{-1}\alpha_0^{-1}) B_0. \tag{7.28}$$

By (7.13) and (7.15), we have

$$\begin{aligned}\alpha_1 + 2\beta_{01}\beta_{11} &= 2\alpha_1, & \alpha_1 + 2\beta_{02}\beta_{12} &= \alpha_1\alpha_0^{-1}(\alpha_0 + \beta_{02}^2),\\ \alpha_2 + 2\beta_{01}\beta_{21} &= 2\alpha_2, & \alpha_2 + 2\beta_{02}\beta_{22} &= \alpha_2\alpha_0^{-1}(\alpha_0 + \beta_{02}^2).\end{aligned} \tag{7.29}$$

Hence, using (7.28),

$$\begin{aligned}(\alpha_1 + 2\beta_{01}\beta_{11}) A_0 = 2\alpha_1 A_0 &= -\alpha_1\alpha_0^{-1}(\alpha_0 + \beta_{02}^2) B_0 = -(\alpha_1 + 2\beta_{02}\beta_{12}) B_0,\\ (\alpha_2 + 2\beta_{01}\beta_{21}) A_0 = 2\alpha_2 A_0 &= -\alpha_2\alpha_0^{-1}(\alpha_0 + \beta_{02}^2) B_0 = -(\alpha_2 + 2\beta_{02}\beta_{22}) B_0.\end{aligned} \tag{7.30}$$

Thus, from (7.27) and (7.28), we obtain

$$A_0 = -k B_0, \qquad A_1 = -k B_1, \qquad A_2 = -k B_2, \tag{7.31}$$

where k is given by

$$k = 1 + \tfrac{1}{4} a_1^{-1}\alpha_0^{-1} = 0.54369, \tag{7.32}$$

by (7.4) and (7.5).

By (6.2) and (6.4), $p^2/\varrho q^2 = \alpha = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2$. Hence if we write

$$p = p_0 + p_1 e_1 + p_2 e_2, \tag{7.33}$$

we find

$$p_0 = (\alpha_0 \varrho q^2)^{\frac{1}{2}}, \qquad p_1 = \tfrac{1}{2}\alpha_1(\varrho q^2/\alpha_0)^{\frac{1}{2}}, \qquad p_2 = \tfrac{1}{2}\alpha_2(\varrho q^2/\alpha_0)^{\frac{1}{2}}. \tag{7.34}$$

Similarly, we write

$$m_1 = m_{01} + m_{11} e_1 + m_{21} e_2, \qquad m_2 = m_{02} + m_{12} e_1 + m_{22} e_2, \tag{7.35}$$

and since $m^2/\varrho q^2 = \beta^2$, we obtain

$$\begin{aligned}m_{01} &= (\varrho q^2)^{\frac{1}{2}}\beta_{01}, & m_{02} &= (\varrho q^2)^{\frac{1}{2}}\beta_{02}, & m_{11} &= (\varrho q^2)^{\frac{1}{2}}\beta_{11},\\ m_{12} &= (\varrho q^2)^{\frac{1}{2}}\beta_{12}, & m_{21} &= (\varrho q^2)^{\frac{1}{2}}\beta_{21}, & m_{22} &= (\varrho q^2)^{\frac{1}{2}}\beta_{22}.\end{aligned} \tag{7.36}$$

By (7.11), (7.13), (7.15), (7.33), (7.34), (7.35), and (7.36), we obtain

$$\begin{aligned}m_1 p &= m_{01} p_0 + e_1(m_{11} p_0 + m_{01} p_1) + e_2(m_{21} p_0 + m_{01} p_2)\\ &= \varrho q^2 \beta_{01}\alpha_0^{\frac{1}{2}} + \varrho q^2 e_1(\beta_{11}\alpha_0^{\frac{1}{2}} + \tfrac{1}{2}\beta_{01}\alpha_1\alpha_0^{-\frac{1}{2}}) + \varrho q^2 e_2(\beta_{21}\alpha_0^{\frac{1}{2}} + \tfrac{1}{2}\beta_{01}\alpha_2\alpha_0^{-\frac{1}{2}})\\ &= \varrho q^2(\alpha_0 + \alpha_1 e_1 + \alpha_2 e_2) = \varrho q^2 \alpha,\\ m_2 p &= m_{02} p_0 + e_1(m_{12} p_0 + m_{02} p_1) + e_2(m_{22} p_0 + m_{02} p_2)\\ &= \varrho q^2 \beta_{02}\alpha_0^{\frac{1}{2}} + \varrho q^2 e_1(\beta_{12}\alpha_0^{\frac{1}{2}} + \tfrac{1}{2}\beta_{02}\alpha_1\alpha_0^{-\frac{1}{2}}) + \varrho q^2 e_2(\beta_{22}\alpha_0^{\frac{1}{2}} + \tfrac{1}{2}\beta_{02}\alpha_2\alpha_0^{-\frac{1}{2}})\\ &= \varrho q^2 \beta_{02}\alpha_0^{-\frac{1}{2}}(\alpha_0 + \alpha_1 e_1 + \alpha_2 e_2) = \varrho q^2 \beta_{02}\beta_{01}^{-1}\alpha.\end{aligned} \tag{7.37}$$

Thus, using (7.31) and (7.37), (7.19) may be written

$$\begin{aligned} u_1 &= -\iota \varrho q^2 k \alpha B (e^{-m_1 x_2} - \beta_{02} \beta_{01}^{-1} k^{-1} e^{-m_2 x_2}) \exp \iota (p x_1 + q t), \\ u_2 &= \varrho q^2 k \alpha B (e^{-m_1 x_2} - k^{-1} e^{-m_2 x_2}) \exp \iota (p x_1 + q t). \end{aligned} \tag{7.38}$$

Now, by (7.11), (7.13), (7.34) and (7.36), when x_1 and x_2 are not large we find

$$\begin{aligned} & B_0^{-1} B \exp [\iota (p_1 e_1 + p_2 e_2) x_1] \exp [-(m_{11} e_1 + m_{21} e_2) x_2] \\ & \quad = 1 + e_1 \{B_1 B_0^{-1} + \iota \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_2\} + \\ & \qquad + e_2 \{B_2 B_0^{-1} + \iota \beta_{21} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{21} (\varrho q^2)^{\frac{1}{2}} x_2\}, \end{aligned} \tag{7.39}$$

and

$$\begin{aligned} & B_0^{-1} B \exp [\iota (p_1 e_1 + p_2 e_2) x_1] \exp [-(m_{12} e_1 + m_{22} e_2) x_2] \\ & \quad = 1 + e_1 \{B_1 B_0^{-1} + \iota \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{12} (\varrho q^2)^{\frac{1}{2}} x_2\} + \\ & \qquad + e_2 \{B_2 B_0^{-1} + \iota \beta_{21} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{22} (\varrho q^2)^{\frac{1}{2}} x_2\}. \end{aligned} \tag{7.40}$$

Hence, from (7.38), using (7.39) and (7.40), we obtain

$$\begin{aligned} & \iota u_1 E \exp [-\iota (\alpha_0 \varrho q^2)^{\frac{1}{2}} x_1 - \iota q t] \\ & = [1 + e_1 \{B_1 B_0^{-1} + \alpha_1 \alpha_0^{-1} + \iota \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_2\} + \\ & \quad + e_2 \{B_2 B_0^{-1} + \alpha_2 \alpha_0^{-1} + \iota \beta_{21} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{21} (\varrho q^2)^{\frac{1}{2}} x_2\}] \exp \{-(\varrho q^2)^{\frac{1}{2}} \beta_{01} x_2\} - \\ & \quad - \beta_{02} \beta_{01}^{-1} k^{-1} [1 + e_1 \{B_1 B_0^{-1} + \alpha_1 \alpha_0^{-1} + \iota \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{12} (\varrho q^2)^{\frac{1}{2}} x_2\} + \\ & \quad + e_2 \{B_2 B_0^{-1} + \alpha_2 \alpha_0^{-1} + \iota \beta_{21} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{22} (\varrho q^2)^{\frac{1}{2}} x_2\}] \exp \{-(\varrho q^2)^{\frac{1}{2}} \beta_{02} x_2\}, \end{aligned} \tag{7.41}$$

and

$$\begin{aligned} & u_2 E \exp [-\iota (\alpha_0 \varrho q^2)^{\frac{1}{2}} x_1 - \iota q t] \\ & = [1 + e_1 \{B_1 B_0^{-1} + \alpha_1 \alpha_0^{-1} + \iota \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_2\} + \\ & \quad + e_2 \{B_2 B_0^{-1} + \alpha_2 \alpha_0^{-1} + \iota \beta_{21} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{21} (\varrho q^2) x_2\}] \exp [-(\varrho q^2)^{\frac{1}{2}} \beta_{01} x_2] - \\ & \quad - k^{-1} [1 + e_1 \{B_1 B_0^{-1} + \alpha_1 \alpha_0^{-1} + \iota \beta_{11} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{12} (\varrho q^2)^{\frac{1}{2}} x_2\} + \\ & \quad + e_2 \{B_2 B_0^{-1} + \alpha_2 \alpha_0^{-1} + \iota \beta_{21} (\varrho q^2)^{\frac{1}{2}} x_1 - \beta_{22} (\varrho q^2)^{\frac{1}{2}} x_2\}] \exp [-(\varrho q^2)^{\frac{1}{2}} \beta_{02} x_2], \end{aligned} \tag{7.42}$$

where

$$E = (\varrho q^2 \alpha_0 k B_0)^{-1}. \tag{7.43}$$

$\alpha_1 \alpha_0^{-1}$ and $\alpha_2 \alpha_0^{-1}$ are given by (7.7), β_{11} and β_{21} by (7.16), and β_{12} and β_{22} by (7.17). From (7.32) we have

$$k^{-1} = 1.8393 \tag{7.44}$$

and hence, using (7.12),

$$\beta_{02} \beta_{01}^{-1} k^{-1} = 0.5438. \tag{7.45}$$

In the case when $e_1 = e_2 = 0$, (7.41) and (7.42) give, on taking the real part of the solutions,

$$\begin{aligned} u_1 &= E^{-1} (e^{-(\varrho q^2)^{\frac{1}{2}} \beta_{01} x_2} - 0.5438 e^{-(\varrho q^2)^{\frac{1}{2}} \beta_{02} x_2}) \operatorname{Sin} (p_0 x_1 + q t), \\ u_2 &= E^{-1} (e^{-(\varrho q^2)^{\frac{1}{2}} \beta_{01} x_2} - 1.8393 e^{-(\varrho q^2)^{\frac{1}{2}} \beta_{02} x_2}) \operatorname{Cos} (p_0 x_1 + q t), \end{aligned} \tag{7.46}$$

as in the classical theory [8].

8. Love Waves

A noteworthy feature of Rayleigh surface waves in classical elasticity theory is the fact that they do not exhibit any horizontal displacements transverse to the direction of propagation. Such horizontal displacements are, however, observed on seismographs. To account for these waves which have the properties (i) of having transverse horizontal movement and (ii) of being transmitted superficially, Love [5] proposed to consider the body of the Earth as being covered by a thin layer of material having mechanical properties different from the material underlying it. Using this model, Love showed that such waves can exist and found that the essential condition for their existence is that the velocity of distortional waves be much less in the layer than in the underlying material.

Here we shall first examine the possibility of having Love-type waves, propagated in a principal direction, in a semi-infinite body which is subjected to a pure homogeneous deformation, without assuming the existence of a superficial layer. It will be seen that this is not possible unless a strong restriction is made on the strain-energy function. Then we shall consider the case when there is a superficial layer and find a condition essentially equivalent to that found by Love for the existence of these waves.

Let the semi-infinite body be subjected to a pure homogeneous deformation (2.1) with which the stresses (2.2) are associated. Superimposed on this deformation is an infinitesimal deformation (2.3). The equations of motion are given by (2.4). The stresses, given by (2.5), may be conveniently written

$$\begin{aligned} \bar{\sigma}_{\alpha\alpha} &= \sum_{\beta=1}^{3} c_{\alpha\beta}\, u_{\beta,\beta}, \\ \bar{\sigma}_{\alpha\beta} &= b_\gamma (\lambda_\alpha^2\, u_{\beta,\alpha} + \lambda_\beta^2\, u_{\alpha,\beta}) \qquad (\alpha \neq \beta \neq \gamma \neq \alpha), \\ b_\gamma &= 2 I_3^{-\frac{1}{2}} (W_1 + \lambda_\gamma^2 W_2). \end{aligned} \tag{8.1}$$

It may be pointed out here that $b_\gamma > 0$ since $\lambda_\alpha^2 b_\beta (\varrho)^{-1}$, $\lambda_\alpha^2 b_\gamma (\varrho)^{-1}$ $\{\alpha \neq \beta \neq \gamma \neq \alpha\}$ are each the square of a velocity with which a plane wave may be propagated along the x_α-direction in an infinite solid of the material when it is subjected to the homogeneous deformation (2.1).

We assume

$$u_1 = u_2 = 0, \qquad u_3 = f(x_2) \exp \iota (p\, x_1 + q\, t), \tag{8.2}$$

where $f(x_2)$ is to be determined such that the motion is confined to the neighborhood of the surface, $x_2 = 0$. The equation of motion not satisfied identically by (8.2) is

$$\lambda_2^2\, b_1\, D^2 f + (\varrho\, q^2 - \lambda_1^2\, b_2\, p^2)\, f = 0, \tag{8.3}$$

where $D = d/d\,x_2$. The admissible solution of (8.3) is

$$f = A\, e^{-m\, x_2},$$

where A is a constant and m^2 is given by

$$\lambda_2^2\, b_1\, m^2 = \lambda_1^2\, b_2\, p^2 - \varrho\, q^2. \tag{8.4}$$

Then

$$u_3 = A\, e^{-m\, x_2} \exp \iota (p\, x_1 + q\, t). \tag{8.5}$$

In writing (8.5) it is assumed that $m^2 > 0$, or equivalently by (8.4),

$$\lambda_1^2 b_2 p^2 > \varrho q^2. \tag{8.6}$$

Otherwise $A = 0$, and hence $f = 0$. From the remark made above, we see that (8.6) means that the square of the velocity of propagation of the surface wave is less than the square of one of the velocities with which a plane wave may be propagated along the x_1 direction in an infinite solid of the same material subjected to the same deformation.

The conditions to be satisfied on the surface of the solid are that it be free of traction both in the case of the deformation (2.1) and in the case of the deformation (2.3). Putting $l_i = (0, -1, 0)$ in (2.9) and using (8.2) we find $\bar{l}_j = (0, 0, 0)$. Hence from (2.7) and (2.10) the conditions to be satisfied on $x_2 = 0$ are

$$\sigma_{22} = 0, \qquad \bar{\sigma}_{23} = 0. \tag{8.7}$$

The first of these gives

$$W_1 + (I_1 - \lambda_2^2) W_2 + \lambda_1^2 \lambda_3^2 W_3 = 0. \tag{8.8}$$

The second gives

$$\lambda_2^2 b_1 m A = 0. \tag{8.9}$$

If $A = 0$ then $u_1 = u_2 = u_3 = 0$. Also, $\lambda_2 = 0$ is impossible. Hence we have two cases to consider: (i) $b_1 = 0$, (ii) $m = 0$. By (8.4) both cases give

$$\lambda_1^2 b_2 p^2 = \varrho q^2. \tag{8.10}$$

In case (i) m may assume any value. When $b_1 = 0$ we have, from (8.1) and (8.7),

$$W_1 = -\lambda_1^2 W_2 = \lambda_1^4 W_3. \tag{8.11}$$

In case (ii)

$$u_3 = A \exp \iota (p x_1 + q t). \tag{8.12}$$

Equation (8.12) is an inadmissible solution since the motion is not confined to the neighborhood of the surface.

Thus we see that only under the very restrictive conditions (8.11) are waves of the required type possible in the material.

We now consider the case when a layer of material, with physical properties different from those of the semi-infinite body, is subjected to a pure homogeneous deformation, with principal extension ratios λ'_α, by forces applied laterally. The layer is then set on the semi-infinite solid which has also been subjected to a pure homogeneous deformation, with principal extension ratios λ_α, by forces applied laterally. We assume that the principal directions for the two pure homogeneous deformations are the same. We neglect gravitational forces. Let the undeformed thickness of the layer be given by H/λ'_2. The interface is given by $x_2 = 0$ and the free surface by $x_2 = -H$. The composite body is now subjected to an infinitesimal deformation.

Symbols with primes refer to the layer and symbols without primes to the underlying body, and we assume that the displacement components are given by (8.2). Then the equations of motion not satisfied identically are

$$\begin{aligned} &\lambda_2^2 b_1 D^2 f + (\varrho q^2 - \lambda_1^2 b_2 p^2) f = 0 \qquad (x_2 \geqq 0), \\ &\lambda_2'^2 b_1' D^2 f + (\varrho' q^2 - \lambda_1'^2 b_2' p^2) f = 0 \qquad (-H \leqq x_2 \leqq 0), \end{aligned} \tag{8.13}$$

and the admissible solutions are

$$\begin{aligned} f(x_2) &= C\,e^{-m x_2} && (x_2 \geqq 0)\,, \\ f(x_2) &= A\cos m' x_2 + B\sin m' x_2 && (-H \leqq x_2 \leqq 0) \end{aligned} \tag{8.14}$$

where

$$\begin{aligned} \lambda_2^2\, b_1\, m^2 &= \lambda_1^2\, b_2\, p^2 - \varrho\, q^2, \\ \lambda_2'^2\, b_1'\, m'^2 &= -\lambda_1'^2\, b_2'\, p^2 + \varrho'\, q^2, \end{aligned} \tag{8.15}$$

A, B, C being constants.

It is essential that $\lambda_1^2 b_2/\varrho > q^2/p^2$, in order that the waves do not penetrate far into the subjacent material. Without loss of generality we may take $m' > 0$. Note that we have assumed that $\lambda_1'^2 b_2'/\varrho' < q^2/p^2$. Later we shall consider the case when $\lambda_1'^2 b_2'/\varrho' > q^2/p^2$.

At the interface $x_2 = 0$, the tangential tractions and the displacement components must be continuous. The surface $x_2 = -H$ must be free of traction. Hence we have the conditions

$$\begin{aligned} &\bar{\sigma}_{23} = \bar{\sigma}_{23}', \qquad u_3' = u_3, \qquad \text{on} \quad x_2 = 0; \\ &\bar{\sigma}_{23}' = 0, \qquad \text{on} \qquad x_2 = -H. \end{aligned} \tag{8.16}$$

We also have $\sigma_{22} = 0$ and $\sigma_{22}' = 0$.

The conditions (8.16) yield

$$\tan m' H = -B/A\,, \qquad \lambda_2^2\, b_1\, m\, C = -\lambda_2'^2\, b_1'\, m'\, B \quad \text{and} \quad A = C. \tag{8.17}$$

On eliminating A, B, C, we obtain the equation

$$\tan m' H = \lambda_2^2\, b_1\, m/(\lambda_2'^2\, b_1'\, m')\,. \tag{8.18}$$

From (8.15), we obtain

$$m^2 = \frac{\lambda_1^2\, b_2}{\lambda_2^2\, b_1}\left(1 - \frac{\varrho\, \lambda_1'^2\, b_2'}{\varrho'\, \lambda_1^2\, b_2}\right) p^2 - \frac{\varrho\, \lambda_2'^2\, b_1'}{\varrho'\, \lambda_2^2\, b_1}\, m'^2. \tag{8.19}$$

In order that m may be real, the right hand side of (8.19) must be positive. Assuming this to be the case, we obtain from (8.18),

$$\tan m' H = \frac{\lambda_2^2\, b_1}{\lambda_2'^2\, b_1'}\left[\frac{\lambda_1^2\, b_2}{\lambda_2^2\, b_1}\left(1 - \frac{\varrho\, \lambda_1'^2\, b_2'}{\varrho'\, \lambda_1^2\, b_2}\right)\frac{p^2}{m'^2} - \frac{\varrho\, \lambda_2'^2\, b_1'}{\varrho'\, \lambda_2^2\, b_1}\right]^{\frac{1}{2}}. \tag{8.20}$$

Then the value of m' corresponding to any value of p must be such that $\tan m'H > 0$ A necessary condition that the right hand side of (8.20) be real is

$$\lambda_1^2\, b_2/\varrho > \lambda_1'^2\, b_2'/\varrho'. \tag{8.21}$$

This condition is analogous to the condition found by LOVE [5]. Equation (8.20) may now be written

$$p\,H = m'\,H\left\{\frac{\varrho'\, \lambda_2'^4\, b_1'^2 \tan^2 m'\,H}{\lambda_2^2\, b_1(\varrho'\, \lambda_1^2\, b_2 - \varrho\, \lambda_1'^2\, b_2')} + \frac{\varrho\, \lambda_2'^2\, b_1'}{(\varrho'\, \lambda_1^2\, b_2 - \varrho\, \lambda_1'^2\, b_2')}\right\}^{\frac{1}{2}}. \tag{8.22}$$

Thus we see that if (8.21) holds there is always a value of m' corresponding to any given value of p. From (8.22) we see that as $m'H$ increases from 0 to $\pi/2$, pH increases from 0 to ∞.

We see from (8.19) that we have dispersion, as in the classical theory. We also have the existence of n nodal planes in the layer if $(2n+1)\pi/2 > m' H > n\pi$.

So far we have assumed $\lambda_1'^2 b_2'/\varrho' < q^2/p^2$. If this is not so, then

$$\begin{aligned} f(x_2) &= C\, e^{-m x_2} \qquad (x_2 \geqq 0), \\ f(x_2) &= A \cosh m' x_2 + B \sinh m' x_2 \qquad (-H \leqq x_2 \leqq 0). \end{aligned} \tag{8.23}$$

The boundary conditions (8.16) give

$$\tanh m' H = B/A, \qquad \lambda_2^2 b_1 m C = -\lambda_2'^2 b_1' m' B \quad \text{and} \quad A = C. \tag{8.24}$$

Hence

$$\lambda_2'^2 b_1' m' \tanh m' H + \lambda_2^2 b_1 m = 0. \tag{8.25}$$

It is necessary that $m > 0$, but since m' tanh $m' H$ is always positive there is no relevant solution in this case.

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The Relation between the Reflection Properties of the Body and the Reflection Properties of its Parts

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Introduction

The nuclear reactor is based on the fact that by joining two subcritical bodies T' and T'' one may obtain a critical or supercritical body T. The neutron transport properties of such a T as a whole are entirely different from those of T' and T'', though the properties of T' and T'' can be considered unchanged. The different behavior of such a T is due to the coupling between T' and T'' caused by the neutron transfer through the contact surface between T' and T''. The reverse procedure is possible as well, *i.e.*, every critical or supercritical body can be split into subcritical parts.

It is natural to ask how the properties of a given body T depend on the properties of its parts T' and T''? This question is the fundamental problem of the reactor theory. It is usually answered only indirectly by determining the life histories of neutrons in the body T by the neutron transport theory, *cf.* [*1*, *2*].

As a consequence, such an answer does not give any transport properties of T expressed explicitly in terms of the analogous properties of T' and T''. Such an explicit answer, however, would be very desirable from the theoretical point of view. This paper presents an attempt to obtain such an explicit connection.

The neutron transport properties of the body observable from the outside are determined by the relation between the distributions of incoming and outgoing neutron currents. These generalized reflection properties of the body give enough information to justify asking how the reflection properties of the body T depend on those of its subcritical parts T' and T''.

The two-group treatment of this problem has been given for the case of simple geometries [*3*], and a similar treatment was recently found useful also in the case of electron diffraction, *cf.* [*4*]. We shall treat it for arbitrary geometries in the frame of the linear transport theory. It is sometimes difficult to understand the theory if the physical model and the mathematical formalism are not separated clearly. Therefore, in Chs. I, II, III, and IV, we shall give the mathematical description of the physical situation and some of its consequences. Later on, in the Appendix, we shall collect a few mathematical derivations.

In Chapter I the reflection properties of a body T are described in linear approximation by an albedo operator A * and a distribution of independent outgoing neutron current q. Two cases are treated: (i) the unsteady case in a finite time interval (§ Ia), and (ii) the steady case (§ Ib).

In Chapter II we study the reflection properties A and q of the body T composed of two parts, T' and T'', which are covered with an absorbing layer of negligible thickness. Assuming quite general reflection properties A', q' and A'', q'' of T' and T'' respectively, in § IIa we derive an inhomogeneous equation of the form $(1-\lambda^4 A' A'')x=y$, where x and y are closely related to the outgoing and incoming neutron currents of T, respectively, and where λ is the transparency factor of the boundary layers. This equation is the mathematical analogue of the physical situation. By studying its properties we shall learn about the reflection properties of the compound body T. The solvability of the corresponding homogeneous equation is the condition that T be critical. By analysis of the inhomogeneous equation we shall construct addition formulae for the reflection properties of T, T', and T''. The results of Chapter II are a generalization of those in [*3*].

Studying the reflection properties of T as a function of the transparency λ, we shall obtain further information about how the reflection properties of T depend on those of T' and T''. In Chapter III, assuming the scattering of neutrons to be uniform, we are able to show that T is subcritical, critical or supercritical according as $\lambda<\lambda_0$, $\lambda=\lambda_0$ or $\lambda>\lambda_0$, where λ_0 is determined explicitly by the albedos A' and A''. The critical condition $\lambda=\lambda_0$ is not directly expressed by the solvability of the critical equation. It might have some practical use for reactor design in the case of albedos A' and A'' which are determined experimentally. For the case $\lambda\to\lambda_0$, $\lambda<\lambda_0$, we shall derive some simple and plain results which are in accordance with the known facts of reactor physics regarding the outgoing neutron currents of T.

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* For some applications of the albedo concept see [*7*], [*8*].

Nearly all necessary mathematical derivations are collected in the Appendix. There are some consequences of the non-negativeness of the neutron current distributions which have an important role in these derivations, *cf.* [2]. Therefore, we shall interpret these distributions as elements of a cone K in the Banach space L_1 chosen so as to match as well as possible the logical structure of our derivations. Since the inhomogeneous equation is found to be a mathematical analogue of the physical problem, the analysis of it is essential for the mathematical part of this paper. This analysis is carried out in two steps: i) assuming the corresponding operator to be additive and to map K into K, and ii) in addition assuming it to be some kind of a uniformly positive operator, *cf.* [2]. Such an analysis of the inhomogeneous equation is done here for the first time, although the analysis of the corresponding homogeneous equation has been given earlier, *cf.* [2, 5].

Chapter I. Reflection properties of a single body

a) Unsteady case in a finite time interval $[0, t_0]$

For simplicity it is assumed that the body T has a time-independent shape and a finite, piecewise smooth, boundary B. This boundary is crossed by neutrons of different velocities whose magnitudes and directions we denote by $v \in [0, c]$ and by unit vectors $\mathbf{s}$, respectively. At each point $\boldsymbol{r} \in B$ the solid angle $\Omega = 4\pi$ is divided into two parts, Ω_- and Ω_+. The part Ω_- is formed by all directions $\mathbf{s}_-$ of the incoming neutrons, while the part Ω_+ is formed by all directions $\mathbf{s}^+$ of the outgoing neutrons (Figure 1).

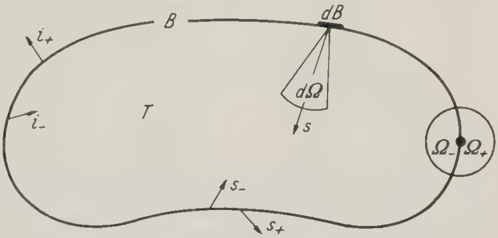

Fig. 1

We adopt the following notation:

i) dB for the neighborhood of the point $\boldsymbol{r}$ on the boundary B.

ii) $d\Omega$ for the element of solid angle around the direction $\boldsymbol{s}$ pointing from $\boldsymbol{r}$.

iii) dJ for the average number of the neutrons passing through dB during the time interval $[t, t+dt]$. The magnitudes of the velocities of these neutrons lie between v and $v+dv$, and their directions belong to $d\Omega$.

iv) $i = i(v, \boldsymbol{r}, \mathbf{s}, t)$ for the neutron current density, where $v \in [0, c]$, $\boldsymbol{r} \in B$, $\mathbf{s} \in \Omega$, and $t \in [0, t_0]$. The relation of i to dJ is given by

$$dJ = i(v, \boldsymbol{r}, \mathbf{s}, t)\, dv\, dB\, d\Omega\, dt. \tag{1.1}$$

The distribution $i(v, \boldsymbol{r}, \mathbf{s}, t)$ depends on the external conditions and on the properties of the body. However, it is always non-negative and such that the total number of neutrons

$$J = \int_0^c \int_B \int_\Omega \int_0^{t_0} i(v, \boldsymbol{r}, \mathbf{s}, t)\, dv\, dB\, d\Omega\, dt \tag{1.2}$$

going through B in the time interval $[0, t_0]$ is finite. Therefore, the unique properties common to all physically possible distributions are assumed to be

$i \geqq 0$ and $J < \infty$. Such distributions form a closed set K in the Banach space

$$I = L_1\big(v \in [0, c],\ \boldsymbol{r} \in B,\ \boldsymbol{s} \in \Omega,\ t \in [0, t_0]\big) \tag{1.3}$$

with the norm $\|i\| = J$ (V. 1, 2, 3)*.

We define the incoming neutron current of T by

$$\begin{aligned} i_- &= i, & \boldsymbol{s} &= \boldsymbol{s}_-, \\ &= 0, & &= \boldsymbol{s}_+ \end{aligned} \tag{1.4}$$

and the corresponding subspace $I_- \subset I$ by

$$I_- = \{f\colon\ f \in I;\ f = 0,\ \boldsymbol{s} = \boldsymbol{s}_+\}.$$

All physically possible distributions i_- form the set

$$K_- = \{i_-\colon\ i_- \in I_-;\ i_- \geqq 0\}.$$

The set of all non-negative functions belonging to the Banach space $I^{(\dots)}_{(\dots)}$ will be denoted henceforth by $K^{(\dots)}_{(\dots)}$.

In an analogous way as for i_- we define the outgoing neutron current by

$$\begin{aligned} i_+ &= i, & \boldsymbol{s} &= \boldsymbol{s}_+, \\ &= 0, & &= \boldsymbol{s}_-. \end{aligned} \tag{1.5}$$

One part of the outgoing neutrons i_+ is due to the incoming neutrons i_-, so that i_+ depends on i_-. In order to have the distribution of the incoming neutrons i_- independent of the outgoing neutrons i_+, we suppose that re-entrance of the outgoing neutrons i_+ is prevented. Hence, the non-convex body T is understood as submerged in an idealized absolutely absorbing medium. If in the body T there are some empty areas which are considered as its outside, they, too, are understood as being filled up by an absolutely absorbing medium.

Let us denote by $q \in K_+$ the outgoing neutrons i_+ corresponding to the incoming current $i_- = 0$. The current q is due to the independent sources inside the body T and to the neutrons which came into the body T during the time interval $t < 0$. Accordingly, the incoming neutrons i_- produce the outgoing neutrons $i_+ - q$.

For a finite time interval $[0, t_0]$ any distribution $i_- \in K_-$ of incoming neutrons is possible. This distribution i_- is transformed by the body T into a uniquely determined distribution $i_+ \in K_+$. This property of the body T determines, by the relations

$$A\,i = A\,i_+ + A\,i_-, \qquad A\,i_+ = 0 \quad \text{and} \quad A\,i_- = i_+ - q, \tag{1.6}$$

the operator A whose domain is K. The operator A is a generalization of the concepts of the albedo and of the transparency, which are known from the transport theory, *cf.* [*1*]; it will be called the albedo. A and q completely describe the reflection properties of the body.

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* In the text we shall denote by (V. ...) those sections of the Appendix that contain the appropriate mathematical tools.

As in the linear transport theory we shall neglect the mutual influence of the neutrons and their influence on the properties of the body. Hence we shall assume:

i) The distribution $A i_-$ of outgoing neutrons corresponding to any distribution i_- is always non-negative. We express this fact by the relation

$$A K \subset K, \tag{1.7}$$

ii) If $i', i'' \in K$, then

$$A(i' + i'') = A i' + A i''. \tag{1.8}$$

An operator having the properties (1.7) and (1.8) will be called additive (V. 4).

In the time interval $[0, t_0]$, as a consequence of the incoming current i_-, $\|i_-\|$ neutrons enter the body, and $\|A i_-\|$ neutrons go out. The ratio $\|A i_-\|/\|i_-\|$ depends on the distribution i_- but is finite for a real body. Its least upper bound is the norm

$$\|A\| = \sup \|A i\|/\|i\| = \sup \|A i_-\|/\|i_-\| \tag{1.9}$$

of the albedo A. Therefore the albedo of a real body is a bounded operator. This follows from its additivity alone (V. 5). Only in the case when $\|A\| > 1$ can there occur distributions i_- such as to cause more outgoing than incoming neutrons in the time interval $[0, t_0]$.

Some of the incoming neutrons will proceed straight through the body, while the rest will be scattered, making it possible for them to be absorbed or multiplied*. The outgoing neutrons produced directly and indirectly by incoming neutrons will be described by additive operators D and S, respectively:

$$A = D + S. \tag{1.10}$$

In the case of an empty body $q = 0$, $S = 0$ and $A = D$.

Let the neutrons i_- form a narrow beam incident at a very small angle on the convex side of the body just after $t = 0$. Nearly all of them will have crossed straight through the body by the time $t = t_0$, and we have $\|D i_-\| \lesssim \|i_-\|$. Hence it follows from the meaning of the norm that for the real body

$$\|D\| = 1,$$

so that from (1.10) we get the estimate

$$1 \leqq \|A\|.$$

The albedo A and the operators S and D are continuous operators; *i.e.*, for all $i_1, i_2 \in K$ we have

$$\|A i_1 - A i_2\| \leqq \|A\| \, \|i_1 - i_2\|, \tag{1.11}$$

and analogously for S and D. We can prove this fact immediately if we extend the operator A to the whole space I by the relation $A(i_1 - i_2) = A i_1 - A i_2$ for all $i_1, i_2 \in K$.

b) Steady case

Let the body T have time-independent properties, and let the incoming currents i_- be stationary. Some of them produce uniquely determined steady outgoing currents i_+. Such currents i_- will be called permissible. The situation on the boundary B is now described by a distribution $i \in K$,

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* In this paper it will be convenient to interpret the multiplication of neutrons as a scattering event.

$$K = \{i\colon\ i \in L_1(v \in [0, c],\ \boldsymbol{r} \in B,\ \boldsymbol{s} \in \Omega);\ i \geqq 0\},$$

as in the unsteady case.

In order to have i_- independent of i_+ we disregard any possible re-entrance of outgoing neutrons i_+. In the unsteady case we distinguished only one type of real bodies. In the stationary case*, however, there are three types of real bodies T. We shall characterize them in accordance with the reactor theory:

i) The subcritical body T: Each distribution $i_- \in K$ of the incoming neutron current is permissible. We describe the reflection properties of subcritical bodies in the same way as in the non-stationary case. However, now the current q is due to the independent sources only, and in the case of elastic scattering $\|A i_-\| = \|i_-\|$ for each $i_- \in K$.

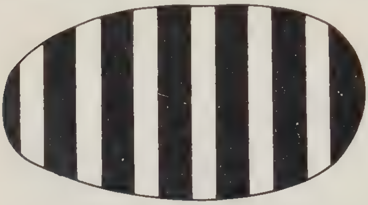

Fig. 2

If the subcritical body T is interpreted as compounded of two subcritical parts T' and T'', their reflection properties A', q', A'' and q'' completely determine the reflection properties A and q of the T.

ii) The critical body T: Each critical body can be imagined as compounded of two subcritical parts T' and T''. For instance, we can obtain subcritical T' and T'' by slicing up the body T and regarding every second slice as a part of T' (Figure 2). Evidently, if the slices are thin enough, they are subcritical by themselves. Consequently, the parts T' and T'' are also subcritical, since neutrons i'_+ and i''_+ are not allowed to re-enter T' and T'', respectively.

The properties of the critical body T are not completely determined by the properties A', A'', q' and q'' of its parts T' and T'', because some of the outgoing neutrons i_+ need not be produced by q', q'' or i_-. The shape of this current, but not its intensity, is determined by the properties of the constituent parts.

iii) The supercritical body T: Some of the distributions $i_- \in K$ are not permissible. Just as in the critical case, we might imagine each supercritical body as compounded of two subcritical parts. But the reflection properties of the supercritical body are completely determined by the reflection properties of these parts.

Since we interpret the critical and supercritical bodies as compounded of two subcritical parts, the properties of such compound bodies grow interesting. They will be considered in Chapters II and III.

Chapter II. Reflection properties of compound bodies

a) Steady case — General considerations

We shall consider the body T as compounded of two finite subcritical bodies T' and T''. Let their boundaries B' and B'' be such that their contact surface $B' \frown B''$ can intersect a straight line at most d times. We consider as the boundary surface B of the compound body T the surfaces B' and B'', excluding the contact surface $B' \cap B''$,

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* It is to be noted that from the physical point of view it would be more adequate to consider the steady case as asymptotic to the unsteady case as $t, t_0 \to \infty$.

$$B = B' \cup B'' - B' \cap B''.$$

The physical situation on the surfaces B' and B'' is given by the neutron currents $i' \in K' \subset I'$ and $i'' \in K'' \subset I''$, respectively. In what follows the spaces I' and I'' will be interpreted as the subspaces of the Banach space

$$I_c = \{i_c\} = L_1(v \in [0, c],\ \boldsymbol{r} \in B' \cup B'',\ \boldsymbol{s} \in \Omega), \tag{2.1}$$

i.e.,

$$I' = \{i_c\colon\ i_c = 0,\ \boldsymbol{r} \in B'' - B'\},$$

and similarly for I''. Let P' and P'' be the projectors associated with the subspaces I' and I'';

$$i' = P' i_c, \quad i'' = P'' i_c \quad \text{and} \quad P' P'' = P'' P'. \tag{2.2}$$

We assume that the properties A_1 and q_1 of T' and A_2 and q_2 of T'' are known and that we do not change them by compounding T' and T'' into T. Before we put T' and T'' together we cover them with a layer of negligible thickness*. Due to the absorption of this layer the intensity of the neutron current crossing the boundary B' or B'' is diminished by a factor $\lambda \in [0, 1]$. Therefore the properties of T' and T'' are changed, and their new properties A', A'', q' and q'' are given by

$$A' = \lambda^2 A_1, \quad A'' = \lambda^2 A_2, \quad q' = \lambda q_1 \quad \text{and} \quad q'' = \lambda q_2,$$

i.e.,

$$i'_+ = q' + A' i'_- \quad \text{and} \quad i''_+ = q'' + A'' i''_-. \tag{2.3}$$

If the boundary layer contains independent neutron sources, additional terms are to be added to the q' and q''.

I' and I'', being interpreted as subspaces of I_c, the ranges of A' and A'' are contained in K_c. It is convenient to extend also their domains K' and K'' to the whole set K_c. For this reason we multiply the right-hand sides of A' and A'' by P' and P'', respectively. These new albedos will be denoted again by A' and A''. It follows that

$$A' = P' A' = A' P' \quad \text{and} \quad A'' = P'' A'' = A'' P''. \tag{2.4}$$

After these changes, which have only formal character, the equations (2.3) remain unchanged, and the albedos A' and A'' become additive operators having the domain K_c.

On the contact surface $B' \cap B''$ the outgoing current of one constituent part is equal to the incoming current of the other one, due to the transfer of neutrons between T' and T'';

$$i'_+ = i''_- \quad \text{and} \quad i'_- = i''_+ \quad \text{for} \quad \boldsymbol{r} \in B' \cap B''. \tag{2.5}$$

As in Chapter I let us assume that on the surface B the outgoing neutrons i'_+ and i''_+ can not re-enter T. Consequently, on the boundary B arbitrary distributions of incoming currents i'_- and i''_- are possible. They define the incoming current i_- of T,

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* To such a layer we attribute absorbing properties only.

$$\begin{aligned} i_- &= i'_-, & \boldsymbol{r} &\in B' - B'', \\ &= i''_-, & \boldsymbol{r} &\in B'' - B', \\ &= 0, & \boldsymbol{r} &\in B' \cap B''. \end{aligned} \tag{2.6}$$

Similarly the outgoing current i_+ of T is defined by

$$\begin{aligned} i_+ &= i'_+, & \boldsymbol{r} &\in B' - B'', \\ &= i''_+, & \boldsymbol{r} &\in B'' - B', \\ &= 0, & \boldsymbol{r} &\in B' \cap B''. \end{aligned} \tag{2.7}$$

The set K of all physically permissible distributions i_- and i_+ belongs to the subspace

$$I = \{i_c\colon\ i_c = 0,\ \boldsymbol{r} \in B' \cap B''\}. \tag{2.8}$$

Let the projector P correspond to the subspace I. Then the equations (2.6) and (2.7) read

$$i_- = P(i'_- + i''_-)$$

and

$$i_+ = P(i'_+ + i''_+). \tag{2.9}$$

The real body contains no absolutely absorbing material. Hence, for later reference, we may assume that for some m

$$P(A' A'')^m i'_+ \neq 0 \quad \text{or} \quad P A''(A' A'')^m i'_+ \neq 0 \quad \text{if} \quad P'' i'_+ \neq 0. \tag{2.10}$$

The incoming current i_- is independent of the outgoing current i_+, due to the assumption that the outgoing neutrons i_+ are prevented from re-entering T. The reflection properties of T are determined by the dependence of i_+ on i_-. We shall express them in terms of the properties A', A'', q' and q'' of T' and T''. In order to do this we shall first determine, using (2.3), the dependence of i'_+ and i''_+ on i_-. For this purpose we must first calculate i'_- and i''_-.

On $B' - B''$ the current i'_- equals the incoming current i_-; on $B' \cap B''$ it equals the outgoing current i''_+. In terms of the projector P' this gives

$$i'_- = P'(i_- + i''_+). \tag{2.11}$$

Similarly we get

$$i''_- = P''(i_- + i'_+). \tag{2.12}$$

From (2.3), (2.11), (2.12) and (2.4) it follows that

$$i'_+ = q' + A'(i_- + i''_+)$$

and

$$i''_+ = q'' + A''(i_- + i'_+). \tag{2.13}$$

Thence, by use of (2.4), we obtain

$$i'_+ = A' A'' P'' i'_+ + y \tag{2.14}$$

with

$$y = q' + A' q'' + A'(1 + A'') i_- \in K'_+.$$

The current i''_+ satisfies an analogous equation. It is sufficient to solve one of these equations only, since the solution of the other one follows then from (2.13).

Equation (2.14) expresses the current i'_+ by its distribution $P'' i'_+$ on $B' \cap B''$. This distribution is determined by the equation

$$(1 - P'' A' A'') P'' i'_+ = P'' y, \qquad (2.15)$$

which is derived by multiplying (2.14) by P''. Equation (2.15) is of fundamental importance for the treatment of the properties of T. Namely, by solving it and using (2.14) and (2.13) we get the outgoing currents i'_+ and i''_+ explicitly. Since $i'_+, y \in K'_+$, we are justified in interpreting (2.15) as an inhomogeneous equation associated with the additive operator $P'' A' A'' P'$ which has the domain $P'' K'_+ = P' K''_-$ (V. 12).

In order to characterize the compound body T, we must answer two questions:

i) What currents i_- are permissible when the body T is a compound of T' and T'', *i.e.*, for what currents i_- does (2.15) have a solution?

ii) Are the properties of T completely determined by the given properties of its parts; *i.e.*, is the solution of (2.15) unique?

Regarding these questions we can say:

i) If (2.15) has a solution at all, the series

$$P'' i'_+ = \sum_{j=0}^{\infty} P'' (A' A'')^j y \qquad (2.16)$$

will be a solution also (V. 12). Consequently, only those currents i_- are permissible for which the series (2.16) converges. If the series (2.16) does not converge even for $i_- = 0$, then the compound body T is physically impossible.

ii) The solution of the (2.15) is unique only if the corresponding homogeneous equation,

$$(1 - \lambda^4 P'' A_1 A_2) P'' i'_+ = 0, \quad \|P'' i'_+\| = 1, \quad P'' i'_+ \in P'' K'_+, \qquad (2.17)$$

has no solution (V. 14). If (2.17) has a solution, the body T is critical. Namely, due to the assumption (2.10), there can be a part of the outgoing neutrons i_+ which is not produced by q', q'' or i_-. Its distribution is not determined by the properties of T' and T'' completely, but up to a positive constant factor. Therefore (2.17) is the critical equation of the compound body T. Thus, only in the critical case do the properties of the compound body fail to be determined by the properties A', A'', q', and q'' of its parts (see § Ib).

Let us derive the series (2.16) from the physical picture for the special steady case with $q'' = 0$, $P'' q' = q'$ and $i_- = 0$: The neutrons q' go through $B' \cap B''$ into T''. One part of them returns (*cf.* the footnote on p. 384) through $B' \cap B''$ into T', and thence one part returns to T'' and so on. From the meaning of the albedos A' and A'' we infer that those neutrons which originate from q' and leave T' through $B' \cap B''$ after being $2n - 1$ times reflected between T' and T'' are given by $P'' (A' A'')^n q'$. Consequently,

$$P'' i'_+ = q' + P'' A' A'' q' + P'' (A' A'')^2 q' + \cdots,$$

which is the same as the series (2.16) adopted for the case $P'' y = q'$. Similarly, one can construct the series (2.16) for the general stationary case as for the non-stationary case.

The series (2.16) gives a clear idea of how the part T'' influences the increase of the outgoing current $P'' i'_+$. The term $P''(A'A'')^n y$ of the series (2.16) represents the neutrons produced by $P''y$ (*cf.* the footnote on p. 384) and entering for the $(n+1)^{\text{th}}$ time the part T''. The norm $\|P''(A'A'')^n P'\|$ is the least upper bound of the ratio $\|P''(A'A'')^n y\|/\|P''y\|$. It has the property that either $\|P''(A'A'')^n P'\| \geqq 1$ for $n=1, 2, \ldots$, or $\lim\limits_{n\to\infty} \|P''(A'A'')^n P'\| = 0$ (V. 6). Of a special interest for us will be the latter case, which holds if, and only if, $\lambda < \lambda_0$ (V. 6),

$$\lambda_0 = \lim_n \|P''(A_1 A_2)^n P'\|^{-\frac{1}{4n}} = \lim_n \|P'(A_2 A_1)^n P''\|^{-\frac{1}{4n}}. \tag{2.18}$$

The constant λ_0, which is the critical value of transparency, satisfies the inequality

$$\lambda_0 > \|P'' A_1 P''\|^{-\frac{1}{4}} \|P' A_2 P'\|^{-\frac{1}{4}}. \tag{2.19}$$

Its connection with the critical compound body will be considered in §§ IIb and IIIb.

If the series (2.16) converges for $i^- = 0$, the compound body T is physically possible and is subcritical, critical or supercritical. Let us consider these cases.

b) The subcritical case

If T is subcritical, (2.15) has a unique solution for each $i_- \in K$, and vice versa. Using the series (2.16), we get from (2.15), (2.14), (2.13) and (2.9) the connection

$$i_+ = q + A i_- \tag{2.20}$$

between the incoming current i_- and the outgoing current i_+. This connection is expressed in terms of the independent outgoing current of T

$$q = (P + A)(q' + q'') \tag{2.21}$$

and of the operator

$$\begin{aligned} A &= PA' + P(1 + A')(1 - A''A')^{-1} A''(1 + A') \\ &= PA'' + P(1 + A'')(1 - A'A'')^{-1} A'(1 + A''), \end{aligned} \tag{2.22}$$

with

$$(1 - A'A'')^{-1} = \sum_{j=0}^{\infty} (A'A'')^j.$$

Operator A in (2.20) is the albedo of the compound body T and is to be interpreted as an additive operator with the domain K. Formulae (2.21) and (2.22) are the addition formulae for the reflection properties. They connect the albedo and the independent outgoing current of T with the same properties of the constituent parts T' and T''.

It is evident that T is subcritical if the transparency λ, determining the coupling between T' and T'', is small enough. Moreover, we prove that the compound body T is surely subcritical if

$$\lambda < \lambda_0, \tag{2.23}$$

for in this case (2.15) always has a unique solution (V. 6, 15). Equations (2.19) and (2.23) show that the compound body is subcritical if

$$\lambda < \|P'' A_1 P''\|^{-\frac{1}{4}} \|P' A_2 P'\|^{-\frac{1}{4}}. \tag{2.24}$$

We shall pause here to give some examples of subcritical compound bodies:

i) Clearly, the compound body T is subcritical if there is no mutual contact between T' and T'', *i.e.*, if $B' \cap B'' = 0$. In this case $P' P'' = 0$, and $P = P' + P''$, and equations (2.19), (2.4), (2.22), (1.6), and (2.21) give

$$\lambda_0 = \infty, \qquad A = A' + A'' \quad \text{and} \quad q = q' + q''.$$

ii) The neutrons are not multiplied in T' and T'', and B' is concave and B'' convex at their contact. Therefore, $\|P' A'' P'\| = 1$, $P'' D' P'' = 0$ and $\|P'' S' P''\| < 1$. The compound body T is always subcritical, since the condition (2.24) now reads $\lambda < \|P'' S' P''\|^{-\frac{1}{4}} > 1$ and is automatically satisfied since $\lambda \in [0, 1]$.

iii) Let it be supposed that there is no scattering in T' and T'', *i.e.*, $A_1 = D_1$ and $A_2 = D_2$. We shall show that

$$P''(D_1 D_2)^n P' = 0 \quad \text{for} \quad n \geqq n_0, \tag{2.25}$$

where n_0 is the smallest integer greater than $\frac{1}{2}(d-1)$. The relation (2.25) follows from the meaning of the operator $P''(D_1 D_2)^n P'$ which relates (for $\lambda = 1$) the neutrons $P' i''_-$ entering T'' through $B' \cap B''$ with the neutrons $P''(D_1 D_2)^n P' i''_-$ leaving T' through $B' \cap B''$. Before the neutrons $P' i''_-$ produce the neutrons $P''(D_1 D_2)^n P' i''_-$ they will have passed n times through T' and T''. Consequently, the surface $B' \cap B''$ is intersected $2n+1$ times by their straight trajectories. Since it was assumed (p. 386) that this can happen at most d times, we get the relation (2.25). From (2.18) and (2.25) it follows that $\lambda_0 = \infty$. Thus the compound body considered is subcritical for all values of λ, and its albedo

$$A = D = P D' + P(1 + D')\,(1 - D'' D')^{-1} D''(1 + D') \tag{2.26}$$

is a finite sum of different products of D' and D''.

The albedo A of the compound body T can be separated into two terms D and S, as has been done in Chapter I, equation (1.10). The neutrons $D i_-$ passing straight through the body T are not affected by the scattering properties S' and S'' of the constituent parts. Therefore the operator D depends on D' and D'' in the same way as the albedo A of the third example. The scattering operator S is determined by

$$S = A - D, \tag{2.27}$$

where A and D are given by (2.22) and (2.26), respectively.

Properties of a subcritical body with an arbitrary number of constituent parts can be also determined from the properties of its parts by (2.20), (2.21) and (2.22). To do so, we join two arbitrary parts into a new part and proceed in this manner until the whole body is built up.

If the outgoing neutrons i_+ do not contribute to the incoming ones i_-, the relation between i_- and i_+ is given by (2.20), with A and q determined by (2.21) and (2.22). Sometimes we are interested in the relation between i_+ and i_- of a non-convex body T, allowing the outgoing neutrons i_+ to contribute to the incoming ones i_-. In this case we consider a compound body whose parts are T and some suitably chosen empty body. We introduce the empty body to create the necessary contribution of the current i_+ to the current i_-.

If it is desired, in the first example above, to take into account the transfer of neutrons between the separated parts T' and T'' or a possible re-entrance of outgoing neutrons, we proceed in a similar way.

c) *The critical case*

If T is critical, (2.17) has one or more linearly independent solutions $(P'' i'_+)_j$, and vice versa. Incoming currents i_- which satisfy $P''(A'A'')^k y \geqq \alpha (P'' i'_+)_j$ for any j, k and some $\alpha > 0$ are not permissible, because the corresponding series (2.16) do not converge. If the incoming current i_- is permissible, from (2.15), (2.14), (2.13) and (2.9) we can determine the corresponding outgoing current i_+ by using the series (2.16). This relation between i_+ and i_- can always be written in the form of equation (2.20), where A is determined by (2.22) and

$$q = (P + A)(q' + q'') + \sum_j \alpha_j P(1 + A') A''(P'' i'_+)_j . \qquad (2.28)$$

The constants α_j are positive and characteristic of T but are not determined by the properties of T' and T''. The operator A of (2.20) can now be interpreted as the albedo of the critical T; its domain is the set of all those $i_- \in K$ for which the series (2.16) converges.

d) *The supercritical case*

If the compound body T is supercritical, for some $i_- \in K$ (2.15) may have a unique solution, and vice versa. If the incoming current i_- is permissible, the outgoing one i_+ is given by (2.20), (2.21) and (2.22). The operator A of (2.20) can now be interpreted as the albedo of the supercritical T, and its domain is those $i_- \in K$ for which the series (2.16) converges. As in the subcritical and the critical case this albedo A can be divided into D and S.

e) *Unsteady case in a finite time interval*

The analysis of the unsteady case is analogous to that of the steady one. In particular, we get relations which are analogous to those of § II a. Then it follows from the physical picture that the equation analogous to (2.15) must always have a unique solution. We shall give a proof of this fact in § III c. Accordingly, the further analysis of the unsteady case is analogous to that of the subcritical case of § II b.

Chapter III. The properties of a compound body when the scattering of neutrons is uniform in some specified sense

a) *Introduction*

In the foregoing chapters the properties of a compound body have been treated under very general assumptions concerning the properties of its parts, as given by equations (1.7), (1.8), (1.10) and (2.25). However, these assumptions are not sufficient to deduce statements about the real compound body T such as:

i) In the unsteady case with a finite time interval each distribution $i_- \in K$ is permissible, and the properties of T are completely determined by the properties of its parts.

ii) In the critical case no physically non-trivial distribution of the incoming current i_- is permissible for steady conditions.

To deduce these and similar statements about a compound body T and to get a better understanding of the influence of the transparency λ on the properties of T, some new properties will be attributed to the albedos A' and A''.

The scattering of neutrons on atomic nuclei is nearly isotropic, and the scattering function of a nucleus is continuous. If the body is large enough, the outgoing scattered neutrons have all possible velocities, and their distribution is nearly independent of the velocity distribution of the incoming neutrons. Besides, the neutrons which arrive together in a body do not leave it at the same time. As a consequence, the scattered neutrons are, in general, distributed more uniformly over the boundary surface and over the time interval than the incoming neutrons. The supplementary mathematical properties which will be attributed to A' and A'' are suggested by just these facts.

b) The steady case

Let us assume henceforth that T' is such that the distribution of the scattered outgoing neutrons $S' i'_- = i'_+(v, \boldsymbol{r}, \boldsymbol{s})$ produced by a narrow beam of monoenergetic incoming neutrons, $i_- = \delta(v - v_0) \cdot \delta(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \delta(\boldsymbol{s} - \boldsymbol{s}_0)$ is a continuous function of $v, \boldsymbol{r}, \boldsymbol{s}, v_0, \boldsymbol{r}_0$ and $\boldsymbol{s}_0$. Let the same be assumed for T''. As a consequence:

i) S' and S'' are integral operators having continuous kernels.

ii) S' and S'' are compact operators. Because of (1.10), (2.3), and (2.25) the operator $P''(A'A'')^{n_0} P'$ is compact, too.

iii) The operator S' satisfies the relation

$$S' i'_- < \|i'_-\| \lambda^2 s', \qquad s' \in K'_+. \tag{3.1}$$

The operator S'' satisfies an analogous relation. From this and from (1.10), (2.3), and (2.25) we get the relation

$$P''(A'A'')^n y < \|P'' y\| \lambda^{4n} s_1, \qquad n \geqq n_0 \quad \text{and} \quad y \in K'_+, \tag{3.2}$$

where $s_1 \in P'' K'_+$ and depends on the exponent n only.

Since distributions of neutron currents are non-negative, there is always a relation opposite to (3.2), namely

$$\varepsilon_y \lambda^{4n} s_2 < P''(A'A'')^n y, \qquad s_2 \in P'' K'_+, \tag{3.3}$$

where ε_y is a positive constant chosen separately for each distribution $P'' y \neq 0$.

If the compound body T is subcritical, it follows from (2.27), (2.22), (2.26), (1.10), (2.25), and (3.1) that its scattering operator S is compact and satisfies a relation analogous to (3.1).

The operator $P''(A'A'')^n P'$ contains $P''(A'A'')^{n_0} P'$ $[n/n_0]$ times as a factor. Due to (1.10) and (2.25), every factor effects at least one scattering. Consequently, the neutrons $P''(A'A'')^n y$ have been scattered at least $[n/n_0]$ times. After each scattering the distribution of the neutrons is smoother and in some sense more uniform. Thus we expect that the greater the number $[n/n_0]$, the more alike will be the distributions $P''(A'A'')^n y$ corresponding to different distributions $P''y$ with norms $\|P''y\|=1$. In what follows we suppose (*cf.* [2]) that with increasing n, the above distributions $P''(A'A'')^n y$ become so much alike that it is possible to choose an integer $n_2 \geqq n_0$ and a distribution $s \in P''K'_+$ such that the relations (3.2) and (3.3) are satisfied for all $y \in K'_+$ if we put $s = s_1 = s_2$ and $n = n_2$ (see V. 18 *et seq.*). Multiplying the relations (3.2) and (3.3) by $P''(A'A'')^m$, we see that analogous relations are valid for all $n > n_2$.

In the case when $\lambda_0 < 1$ the reflection properties of T depend on the transparency λ as follows:

i) for $\lambda_0 < \lambda \leqq 1$ the compound body is supercritical,

ii) for $\lambda = \lambda_0$ the compound body is critical,

iii) for $0 \leqq \lambda < \lambda_0$ the compound body is subcritical.

i) If $\lambda > \lambda_0$, only those currents q', q'' and i_- are permissible for which $P''y=0$, since only if $P''y=0$ does (2.15) have a solution. This solution is $P''i'_+=0$, and it is unique (V. 19, 6, 23). With $P''y=0$ and $P''i'_+=0$ we get from (2.14) and (2.13) the relations

and
$$\begin{aligned} i'_+ &= y \\ i''_+ &= q'' + A''i_- . \end{aligned} \tag{3.4}$$

The condition $P''y=0$ is satisfied only if the currents q', q'' and i_- do not cause any transfer of neutrons through $B' \cap B''$ from T' into T''. As a consequence, a supercritical real compound body T without empty areas is physically possible only if $i_-=0$, $q'=0$ and $q''=0$. Indeed, if $i_- \neq 0$, since the scattering of neutrons is nearly isotropic, some of the incoming neutrons will be scattered through $B' \cap B''$ from T' into T''. Besides, if a part of such a compound body has an independent outgoing neutron current, its distribution is always such that there is a current of neutrons through any part of the boundary surface. Therefore the domain of the albedo A of such a supercritical T is an empty set and $q=0$ (*cf.* § II d). Thus, a supercritical compound body with $q' \neq 0$, $q'' \neq 0$ or $i_- \neq 0$ is not possible unless it contains suitable empty areas.

ii) If $\lambda = \lambda_0$, only those q', q'' and i_- are permissible for which $P''y=0$ (V. 19, 6). Further, the critical equation (2.17) has a unique solution $(P''i'_+)_0$ (V. 23, 25). With $P''y=0$ we get from (2.14) and (2.13) the relations

and
$$\begin{aligned} i'_+ &= y + \alpha A'A''(P''i'_+)_0 \\ i''_+ &= q'' + A''(i_- + \alpha(P''i'_+)_0), \end{aligned} \tag{3.5}$$

where α is a positive constant, characteristic of the compound T and not determined by the properties of T' and T''.

In the case of a real compound body T without empty areas, the equation (3.5) can be simplified by taking into account the physical meaning of the condition $P''y=0$. We get for the domain of the albedo A the empty set (*cf.* § II c) and for the independent outgoing current q the expression

$$q = \alpha P(1+A')A''(P''i'_+)_0 . \tag{3.6}$$

iii) If $\lambda<\lambda_0$, the compound body T is subcritical, as we have shown in § IIb, where also the addition formulae for the reflection properties were given. Nevertheless, the body T under consideration is subcritical if and only if $\lambda<\lambda_0$.

The outgoing current $P'' i'_+$ satisfies for all $y \in K'_+$ the relations

$$P'' i'_+ > \sum_{j=0}^{n_2-1} P''(A' A'')^j y + \varepsilon_y \frac{(\lambda/\lambda_0)^{4 n_2}}{1-(\lambda/\lambda_0)^4} (P'' i'_+)_0$$

and

$$P'' i'_+ < \sum_{j=0}^{n_2-1} P''(A' A'')^j y + \frac{(\lambda/\lambda_0)^{4 n_2} \|P'' y\|/\varepsilon_0}{1-(\lambda/\lambda_0)^4} (P'' i'_+)_0, \tag{3.7}$$

where $\lambda \in (0, \lambda_0)$, and ε_0 is the ε_y of the relation (3.3) with $y=(P'' i'_+)_0$ (V. 24).

Let us now consider the changes of the reflection properties of the subcritical T when λ tends towards λ_0. It follows immediately from (3.7) that the outgoing current $P'' i'_+$ produced by any $P'' y \neq 0$ becomes arbitrarily great, provided λ is close enough to λ_0. Moreover, if $\lambda \to \lambda_0$ its shape $P'' i'_+/\|P'' i'_+\|$ approaches the solution of the critical equation (2.17) (V. 25, 22); *i.e.*,

$$P'' i'_+/\|P'' i'_+\| \to (P'' i'_+)_0$$

if

$$\lambda \to \lambda_0 \quad \text{and} \quad P'' y \neq 0. \tag{3.8}$$

The same holds for i'_+ and i''_+.

If $\lambda_0=1$, the first case is impossible. If $\lambda_0<1$ the second case is impossible too, since $\lambda \in [0, 1]$. Therefore, the critical condition for the compound body considered is

$$\lambda_0 \leqq 1 \quad \text{and} \quad \lambda=\lambda_0. \tag{3.9}$$

After the foregoing considerations the operation of a reactor can be interpreted as follows:

Due to spontaneous fission of nuclei the real subcritical body always has independent sources q' and q'' of very low intensity. If λ is close enough to the critical value λ_0, these sources can still produce an arbitrarily large outgoing current i_+ even if $i_-=0$. Its shape $i_+/\|i_+\|$ can be approximated by the solution of the critical equation; *i.e.*, if $\lambda \lesssim \lambda_0$, then

$$i_+/\|i_+\| \approx P(1+A') A''(P'' i'_+)_0/\|P(1+A') A''(P'' i'_+)_0\|. \tag{3.10}$$

This relation is derived from (2.14), (2.13), (2.9), (3.8), and (2.10), taking account of (3.7).

c) The unsteady case in a finite time interval

Let us assume the following properties of the albedos A' and A'':

i) The distribution i'_+ of the outgoing neutrons at a certain moment is not affected by the distribution i'_- of the incoming neutrons at a later time; *i.e.*, the albedo A' satisfies the causality condition

$$A' i'_- = 0 \quad \text{for all} \quad t \in [0, t_1],$$

if

$$i'_- = 0 \quad \text{for all} \quad t \in [0, t_1]. \tag{3.11}$$

The same holds for the albedo A'' and for S', S'', D' and D''.

ii) The scattering operators S' and S'' satisfy relations analogous to (3.1).

It is evident that any real body satisfies the causality condition (3.11). The second property assumed is very general and can be attributed to every real body without loss of generality. From these two assumptions it follows that the operator $P'' A_1 A_2 P'$ satisfies the causality condition and a relation which is analogous to (3.2)*. As a consequence, $\lambda_0 = \infty$ (V. 11), and the condition (2.23) tells us that, for all values of λ, (i) the properties of T are completely determined by the properties of its parts, and (ii) any incoming current $i_- \in K$ is permissible.

Chapter IV. Resumé

The steady reflection properties of a subcritical body have been described by means of a linear albedo operator A and a distribution q of independent outgoing neutrons. It has been shown that the boundedness of A follows from its linearity and from the non-negativeness of the distribution functions of the neutron currents. The same holds for the unsteady case in a finite time interval.

We have considered some general properties of a compound body whose parts T' and T'' have been covered with a negligibly thin layer of absorbing material. It has been shown that in the steady case such a compound body is completely characterized by the properties A', A'', q' and q'' of its parts, with the exception of the critical case. The albedo A of a compound body and its independent outgoing neutron current q were expressed by A', A'', q' and q'' in the subcritical, critical and supercritical cases. However, in the critical case the independent outgoing current of a compound body is not completely determined by the properties of T' and T''. The compound body was found to be subcritical if the transparency λ of the boundary layer of its parts is lower than some critical value λ_0; this value is determined by the albedos of the uncovered parts T' and T''.

Assuming that the scattering of neutrons is uniform in some specified sense, we obtained a clearer picture of the relation between the properties of the compound body and the transparency λ of the boundary layer. It was possible to deduce the following statements about the compound body:

i) In the unsteady case in a finite time interval the compound body is completely characterized by the properties of its parts, and any distribution of incoming current is permissible.

ii) In the steady case, there are the necessary and sufficient conditions for the compound body to be subcritical, critical or supercritical. They are expressed in terms of transparency of the boundary layer and of the albedos of the uncovered constituent parts. Thus we obtained the critical condition, which is not connected directly with the solvability of a critical equation.

iii) If, in the steady case, the transparency λ of the boundary layer tends towards the value λ_0, the compound body is growing critical; *i.e.*, the outgoing current becomes infinite with $(\lambda_0 - \lambda)^{-1}$, and its distribution approaches the critical one. The critical distribution is determined by the critical equation up to a positive constant factor.

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* We note that, due to the causality condition, the operator $P'' A_1 A_2 P'$ can satisfy a relation of the type (3.3) only in the trivial case when $s_2 = 0$.

Chapter V. Appendix

Introduction

The mathematical formalism used in this paper for the description of the physical phenomena consists of

i) a set K of all non-negative functions belonging to a space L_1,

ii) linear operators A which map K into K. They will be called additive operators.

The set K will be treated as an abstract closed subset of a Banach space, characterized by four properties. If K is a set of all non-negative numbers, these properties are:

i) a non-negative number multiplied by a positive number remains non-negative,

ii) and iii) the sum of two non-negative numbers is a non-negative number which is never smaller than any of the summands,

iv) a bounded infinite series of non-negative numbers is convergent.

The boundedness of an additive operator and some usual properties of its powers will be proved first. Then the estimate $\|AB\| \leqq \|A\|\,\|B\|$ will be generalized to prove that an operator which satisfies the causality condition and the relation $Ax < \|x\|\,a$ for some $a \in K$, and for all $x \in K$, is quasi-nilpotent.

The properties of the inhomogeneous equation $(1-\lambda A)x = y$, where $\lambda > 0$, and $x, y \in K$, are of decisive importance for the physical problem treated. It will be shown that:

i) The solution of the inhomogeneous equation is unique if, and only if, the corresponding homogeneous equation has no non-trivial solution.

ii) If there is a solution of the inhomogeneous equation, the Neumann series is also a solution.

iii) The Neumann series converges uniformly if λ is smaller than the spectral radius λ_0 of A.

Special attention is given to the spectral properties of a continuous compact operator A which, for all $x \in K$ and some $a \in K$, satisfies the relation $\varepsilon(x)\,a < Ax < \|x\|\,a$ where $\varepsilon(x) > 0$ if $x \neq 0$. It will be shown that:

i) The corresponding inhomogeneous equation has a unique solution x for all $y \in K$ if $\lambda < \lambda_0$. But, there is no solution for any $y \in K$, $y \neq 0$, if $\lambda \geqq \lambda_0$.

ii) The corresponding homogeneous equation has a non-trivial solution x_0 only for $\lambda = \lambda_0$. This solution is determined up to a positive constant factor.

iii) The solution x of the inhomogeneous equation, corresponding to any $y \neq 0$, has the property that $\|x\| \to \infty$ and $x/\|x\| \to x_0/\|x_0\|$ for $\lambda \to \lambda_0$, $\lambda < \lambda_0$.

1. Definition. Let K be a closed set in a Banach space B. Let the elements of this set K have the following properties:

1. if $x \in K$ and if $\alpha > 0$, then $\alpha x \in K$,
2. if $x, y \in K$, then $x + y \in K$,
3. if $x, y \in K$, then $\|x+y\| \geqq \|x\|$,
4. if the subset $\{x_i\}_0^\infty \subset K$ and if $\lim\limits_n \left\|\sum\limits_{i=0}^{n} x_i\right\| < \infty$, then the series $\sum\limits_{i=0}^{\infty} x_i$ converges, *i.e.*, $\sum\limits_{i=0}^{\infty} x_i \in K$.

2. Lemma. 1. *If* $x, -x \in K$, *then* $x=0$.

2. *If* $x, y, x-y \in K$, *then* $\|x\| \geqq \|y\|$.

Proof. 1. If $x, -x \in K$, it follows from (1.3) that $0=\|x-x\| \geqq \|x\|$; henc $x=0$.

2. If $x, y, x-y \in K$, then $x=y+z$, where $z \in K$, and we get from (1.3) th relation $\|x\|=\|y+z\| \geqq \|y\|$. Q.E.D.

3. Example. Let B be $L_p(0,1)$. Then an example of K is the set

$$K_p = \{r(t) \exp(i\varphi(t)) : r(t) \in L_p(0,1);\ |\varphi(t)-\alpha(t)| \leqq \beta(t)\}, \qquad (1$$

where $\alpha(t)$ and $\beta(t)$ are real functions such that $0 \leqq \alpha(t) < 2\pi$ and $0 \leqq \beta(t) \leqq \pi/4$ for $t \in [0,1]$.

For any value t_1 of the argument t, the values of all functions of the set K form the sector $|\arg z - \alpha(t_1)| \leqq \beta(t_1)$ in a complex plane. Therefore, the element of the set K_p have the properties (1.1), (1.2) and (1.3). It is easy to prove that the condition (1.4) is satisfied too. This can be done by introducing an auxiliary coordinate system with the coordinate vectors $e_1 = \exp(\alpha(t)-\pi/4)$ and $e_2 = \exp(\alpha(t)+\pi/4)$ and by making use of the inequality

$$(x+y)^p \geqq x^p + y^p, \qquad x, y \geqq 0, \qquad p \geqq 1,$$

and of the completeness of the space L_p.

Analogous examples of K exist in the space l_p, too.

4. Definition. In this paper, the operator A with the domain K is calle additive if

$$A K \subset K \qquad (1$$

and if

$$A(x+y) = Ax + Ay, \qquad x, y \in K. \qquad (2$$

5. Lemma. *An additive operator* A *is homogeneous and bounded*, i.e.,

$$A(\lambda x) = \lambda A x, \qquad \lambda \geqq 0, \qquad x \in K, \qquad (1$$

and

$$A(x/\|x\|) < \text{const.}, \quad \textit{for all} \quad x \in K. \qquad (2$$

Thus A *has a norm defined by*

$$\|A\| = \sup_{x \in K} \|Ax\|/\|x\|. \qquad (3$$

Proof. 1. It follows immediately from (4.2) and (1.1) that the relation (5.1 holds for all non-negative rational values λ_r of λ.

2. There is an infinite sequence

$$\{x_n:\ x_n \in K;\ \|x_n\| = 1;\ \|A x_n\| > n^3;\ n = 1, 2, 3, \ldots\}, \qquad (4$$

if A is not bounded. Now, considering (4.2), (1.3), (5.1), and (5.4), and usin the series

$$\sum_{n=1}^{\infty} x_n/n^2 \in K,$$

we have the contradiction

$$\infty > \left\| A \sum_{n=1}^{\infty} x_n/n^2 \right\| \geqq \lim_n \|A(x_n/n^2)\| \geqq \lim_n n = \infty.$$

Hence any additive operator A is bounded.

3. Since A is bounded, we have for any $\lambda > 0$

$$A(\lambda\, x) = \lim_{\lambda_r < \lambda,\ \lambda_r \to \lambda} \{A[(\lambda - \lambda_r)\, x] + \lambda_r\, A\, x\} = \lambda\, A\, x. \qquad \text{Q.E.D.}$$

6. Alternative. *If A is an additive operator, the following alternative holds: either*

$$\|A^n\| \geqq 1 \quad \textit{for all} \quad n = 1, 2, \ldots \tag{1}$$

or

$$\lim_{n \to \infty} \|A^n\| = 0. \tag{2}$$

Furthermore, the limit

$$\lambda_0 = \lim_n \|A^n\|^{-\frac{1}{n}} \geqq \|A\|^{-1} > 0 \tag{3}$$

exists, and the operator λA $(\lambda \geqq 0)$ satisfies (6.1) *if $\lambda \geqq \lambda_0$, or* (6.2) *if $\lambda < \lambda_0$. We shall call the number λ_0 the spectral radius of the operator A.*

Let A and B be additive operators. Then

$$\lim_n \|(A\, B)^n\|^{-\frac{1}{n}} = \lim_n \|(B\, A)^n\|^{-\frac{1}{n}}. \tag{4}$$

Proof. 1. Let be $\|A^{n_1}\| = q < 1$. Then we choose an integer N such that $q^N \|A\|^i < 1$ for all $i \in [0, n_1 - 1]$. Thence we get for the power A^n $(n = n_1(N+m) + i$, $m = 0, 1, 2, \ldots)$ the estimate

$$\|A^n\| \leqq \|A^{n_1}\|^{(N+m)} \|A\|^i < q^m$$

having the limit (6.2) as a consequence.

2. The operator λA satisfies the relation (6.1) if, and only if, $\lambda \geqq \sup_{n=1,2,\ldots} \|A^n\|^{-\frac{1}{n}}$. It is known [*6*] that

$$\sup_n \|A^n\|^{-\frac{1}{n}} = \lim_n \|A^n\|^{-\frac{1}{n}}.$$

3. The relation (6.4) follows from the estimate

$$\|(A\, B)^n\|^{-\frac{1}{n}} \geqq (\|A\|\, \|B\|)^{-\frac{1}{n}} \|(B\, A)^{n-1}\|^{-(1-\frac{1}{n})/n-1}$$

and from the one we get by interchanging A and B.

7. Definitions. The direct sum of N Banach spaces $E_i = \{x_i\}$, $i = 1, 2, \ldots, N$, is denoted by

$$B_p^N = \{x\colon\ x = (x_1, x_2, \ldots, x_N)\}. \tag{1}$$

The norm of x is given by

$$\|x\| = \left(\sum_{i=1}^{N} \|x_i\|^p\right)^{1/p}, \qquad p \geqq 1. \tag{2}$$

The projectors P_i are defined by

$$P_i\, x = (\delta_{i1}\, x_1,\ \delta_{i2}\, x_2, \ldots, \delta_{iN}\, x_N). \tag{3}$$

We have

$$\sum_{i=1}^{N} P_i = 1. \tag{4}$$

A mapping of B_p^N into a space

$$l_p^N = \left\{x'\colon\ x' = (x_1', x_2', \ldots, x_N'),\ x_i' \in (-\infty, \infty);\ \|x'\| = \left(\sum_{i=1}^{N} |x_i'|^p\right)^{1/p}\right\} \tag{5}$$

is defined such that to each $x \in B_p^N$ corresponds an $x' \in l_p^N$ having the components

$$x'_i = \|P_i\, x\|. \tag{6}$$

This mapping is isometric; *i.e.*,

$$\|x\| = \|x'\|. \tag{7}$$

Let K_p^N be a closed set in B_p^N which has the properties (1.1), (1.2), (1.3), (1.4) and

$$P_i\, K_p^N < K_p^N, \qquad i = 1, 2, \ldots, N. \tag{8}$$

Further, let A be an additive operator with the domain K_p^N. Then an associated matrix A' is defined by

$$A'_{ij} = \|P_i\, A\, P_j\|, \qquad 1 \leqq i, j \leqq N. \tag{9}$$

A' is to be interpreted as a linear operator with the domain l_p^N.

8. Theorem. *Let A_k $(k = 1, 2, \ldots, m)$ be additive operators with the domain K_p^N, and let A'_k be the associated operators with the domain l_p^N. Then, for any integer m, we have the estimate*

$$\left\| \prod_{k=1}^{m} A_k \right\| \leqq \left\| \prod_{k=1}^{m} A'_k \right\|. \tag{1}$$

Proof. From (7.6), (7.4) and (7.9) we get the estimate

$$(A\, x)'_i = \left\| \sum_{j=1}^{N} P_i\, A\, P_j\, (P_j\, x) \right\| \leqq \sum_{j=1}^{N} A'_{ij}\, x'_j = (A'\, x')_i .$$

By its use we get from (7.7) and (7.5) the estimate

$$\left\| \prod_{k=1}^{m} A_k\, x \right\| \leqq \left\| A'_m \Bigl(\prod_{k=1}^{m-1} A_k\, x \Bigr)' \right\| \leqq \left\| A'_m\, A'_{m-1} \Bigl(\prod_{k=1}^{m-2} A_k\, x \Bigr)' \right\| \leqq \cdots$$
$$\leqq \left\| \prod_{k=1}^{m} A'_k\, x' \right\| \leqq \left\| \prod_{k=1}^{m} A'_k \right\| \|x\|.$$

This proves (8.1).

9. Lemma. *Let T be the linear matrix operator with the domain l_p^N and having the elements*

$$\begin{aligned} T_{ij} &= 1, \qquad j \leqq i, \\ &= 0, \qquad j > i, \quad 1 \leqq i, \quad j \leqq N. \end{aligned} \tag{1}$$

Then its n^{th} power, T^n, is given by

$$\begin{aligned} (T^n)_{ij} &= \binom{i - j + n - 1}{n - 1}, \qquad j \leqq i, \\ &= 0, \qquad\qquad j > i \end{aligned} \tag{2}$$

and satisfies the relation

$$\|T^n\| \leqq N \max_{i,j} (T^n)_{ij} < (n + N)^N. \tag{3}$$

Proof. 1. When $n = 2$, the relation (9.2) can be checked directly. For an arbitrary integer n we prove (9.2) by complete induction. It follows from (9.1) that

$$(T^n)_{ij} = (T\, T^{n-1})_{ij} = \sum_{k=1}^{N} T_{ik}\, (T^{n-1})_{kj} = \sum_{k=1}^{i} (T^{n-1})_{kj} .$$

From this expression and from (9.2) with the exponent $n-1$, the relation (9.2) with the exponent n follows immediately, provided $j>i$. For $j\leqq i$, the same follows from

$$(T^n)_{ij}=\sum_{k=j}^{i}\binom{k-j+n-2}{n-2}=\binom{i-j+n-1}{n-1}.$$

2. The estimate (9.3) follows from (9.2) and from Hoelder's inequality. Q.E.D.

10. Theorem. *Let A be an additive operator with the domain K_p^N, whose associated operator A' has the properties*

$$A'_{ij}=0, \qquad j>i,$$

and

$$\max_{i,j} A'_{ij}=q<1. \tag{1}$$

Then

$$\lim_n \|A^n\|=0.$$

Proof. From (8.1), (10.1) and (9.3) follows

$$\lim_n \|A^n\|\leqq \lim_n \|(A')^n\|\leqq \lim_n q^n\,\|T^n\|\leqq \lim_n q^n\,(n+N)^N=0.$$

11. Example. The Banach space $L_p(0,1)=\{x(t)\}$ can be interpreted as a space B_p^N which is a direct sum of an arbitrary number N of the spaces

$$E_i=L_p\left(\frac{i-1}{N},\,\frac{i}{N}\right), \qquad i=1,2,\ldots,N. \tag{1}$$

In this case, the set K_p given by (3.1) is an example of the set K_p^N. As an example for Theorem 10, we consider the additive operator S having the domain K_p and satisfying:

i) the causality condition; *i.e.*, if

$$x=0 \quad \text{for all} \quad t\in[0,t'], \qquad t'\in[0,1], \qquad x\in K_p, \tag{2}$$

then

$$S\,x=0 \quad \text{for all} \quad t\in[0,t']; \quad \text{and}$$

ii) the relation

$$\|x\|\,s-S\,x\in K_p \tag{3}$$

for all $x\in K_p$ and some $s\in K_p$.

Namely, if N is large enough, the associated operator S' has the properties (10.1).

Proof. 1. It follows from (7.9), (7.3), (11.1) and (11.2) that

$$S'_{ij}=\|P_i\,S\,P_j\|=0, \qquad i<j.$$

2. From (7.9), (5.3), (7.3), (7.2), (2.2), and (11.3) we get the estimate

$$S'_{ij}=\sup\|P_i\,S\,(P_j\,x)\|/\|P_j\,x\|\leqq\|P_i\,s\|,$$

and with $|s|\in L_p(0,1)$ it follows from (10.1) and (11.1) the relation

$$\lim_{N\to\infty} q\leqq\lim_N\max_i\|P_i\,s\|=\lim_N\max_i\left(\int_{\frac{i-1}{N}}^{i/N}|s(t)|^p\,dt\right)^{1/p}=0. \qquad \text{Q.E.D.} \tag{4}$$

Let D be an additive and nilpotent operator,

$$D^n = 0, \quad n \geqq n_0, \tag{5}$$

which satisfies the causality condition (11.2) and has the domain K_p. Then the operator $A = S + D$ is quasi-nilpotent; *i.e.*,

$$\lim_n \|\lambda^n A^n\| = 0$$

for each $\lambda \in [0, \infty)$.

Proof. Since S and D satisfy the causality condition, the operator $(\lambda A)^n$ satisfies this condition, too. From (11.5) and (11.3) we infer that the operator $(\lambda A)^{n_0}$ satisfies a relation which is analogous to (11.3). As a consequence there exists, for each $\lambda \in [0, \infty)$, an integer N such that $((\lambda A)^{n_0})'$ has the properties (10.1). Hence $\|(\lambda A)^{m n_0}\| < 1$ if the integer m is great enough. Therefore, in accordance with Alternative 6, $\lim_n \|(\lambda A)^n\| = 0$ for any value of λ. Q.E.D.

12. Lemma. *Let*

$$(1 - A)\, x = y, \qquad y \in K, \tag{1}$$

be an inhomogeneous equation associated with an additive operator A having the domain K. Then the Neumann series

$$x_{ap} = \sum_{i=0}^{\infty} A^i y \tag{2}$$

is a solution of (12.1) *if, and only if,* $x_{ap} \in K$, cf. [*6*].

Proof. 1. It is evident that this condition is necessary, since K is the domain of A.

2. If $x_{ap} \in K$,

$$\left\| \sum_{i=0}^{\infty} A^i y - A \sum_{i=0}^{\infty} A^i y - y \right\| \leqq (1 + \|A\|) \lim_n \left\| \sum_{i=n}^{\infty} A^i y \right\| = 0. \qquad \text{Q.E.D.}$$

13. Theorem. *Let $x_1 \in K$ be a solution of* (12.1). *Then $x_{ap} \in K$ and is, therefore, a solution of* (12.1). *The difference*

$$x_1 - x_{ap} = \lim_n A^n x_1 \in K$$

is a solution of the homogeneous equation

$$(1 - A)\, x = 0, \tag{1}$$

corresponding to (12.1).

Proof. 1. Writing the solution x_1 in the form

$$x_1 = y + A\,y + A^2 y + A^3 y + \cdots + A^n y + A^{n+1} x_1, \tag{2}$$

and taking account of (2.2), we get

$$\lim_n \left\| \sum_{i=0}^{n} A^i y \right\| \leqq \|x_1\| < \infty.$$

Thence and from (1.4) it follows that $x_{ap} \in K$, and as a consequence of Lemma 12, the series x_{ap} is a solution of (12.1).

2. From (13.2) we get the relation

$$\|A^{m+n} x_1 - A^n x_1\| = \left\| \sum_{i=n}^{m+n-1} A^i y \right\|$$

which shows, due to the convergence of the series x_{ap}, that the sequence $\{A^n x_1\}_0^\infty$ is a Cauchy sequence. Since K is a closed set, it follows that $\lim\limits_n A^n x_1 \in K$.

Using (13.2), we prove that

$$\|x_1 - x_{ap} - \lim_i A^i x_1\| \leqq \lim_n \|A^n x_1 - \lim_i A^i x_1\| + \lim_n \left\| \sum_{i=n}^{\infty} A^i y \right\| = 0 .$$

It follows from $x_1 - x_{ap} \in K$ and from

$$y = (1 - A)\, x_1 = (1 - A)\, x_{ap} + (1 - A)\,(x_1 - x_{ap}) = y + (1 - A)\,(x_1 - x_{ap})$$

that the difference $x_1 - x_{ap}$ is a solution of (13.1). Q.E.D.

14. Theorem. *If the inhomogeneous equation* (12.1) *has a solution, then this solution is unique if, and only if, the corresponding homogeneous equation* (13.1) *has no non-trivial solution.*

Proof. 1. It is evident that the stated condition is necessary.

2. Let the homogeneous equation (13.1) have no non-trivial solution, and let x_1 be a solution of (12.1). Then, as a consequence of Theorem 13, the series x_{ap} is a solution of (12.1), and the difference $x_1 - x_{ap}$ is a solution of (13.1). Therefore $x_1 - x_{ap} = 0$, which proves that the stated condition is sufficient. Q.E.D.

15. Theorem. *The series x_{ap} is uniformly convergent if it satisfies the Cauchy condition of convergence uniformly for all elements of the set $\{y\colon\, y \in K;\ \|y\| = 1\}$. The condition $\lim\limits_n \|A^n\| = 0$ is necessary and sufficient for the uniform convergence of the series x_{ap}. If this condition is satisfied, the series x_{ap} is the unique solution of* (12.1) (*cf.* [6]).

Proof. 1. It follows from Alternative 6 and from the definition of the uniform convergence that the stated condition is necessary.

2. If $\lim\limits_n \|A^n\| = 0$, we have

$$\begin{aligned} \left\| \sum_{i=n}^{n+m} A^i y \right\| &\leqq \|y\| \sum_{i=n}^{\infty} \|A^i\| \leqq \|y\| \sum_{j=[n/n_1]}^{\infty} \sum_{i=0}^{n_1} \|A^{j n_1 + i}\| \\ &\leqq \|y\| \sum_{j=[n/n_1]}^{\infty} \sum_{i=0}^{n_1} \|A^{n_1}\|^j \|A\|^i = \|y\|\, q^{[n/n_1]} \Big(\sum_{i=0}^{n_1} \|A\|^i \Big) \Big/ (1 - q), \quad m > 0,\ n > 0, \end{aligned}$$

where n_1 is chosen such that

$$q = \|A^{n_1}\| < 1 .$$

This estimate tells us that the series x_{ap} satisfies uniformly the Cauchy test of convergence. According to Lemma 12, the series x_{ap} is a solution of (12.1).

3. Let x be a solution of (13.1). Then

$$(1 - A^n)\, x = 0, \qquad n = 0, 1, 2, 3, \ldots .$$

If $\lim\limits_n \|A^n\| = 0$, it follows that $\|x\| \leqq \|x\| \lim\limits_n \|A^n\| = 0$, and according to Theorem 14 the series x_{ap} is the unique solution of (12.1).

16. Theorem. *Let A be an additive operator with the domain K, and let $(1-\lambda A)$, $\lambda \geqq 0$, be an operator with the domain $\{x\colon x \in K;\ (1-\lambda A)\,x \in K\}$. Then when $\lambda < \lambda_0$, there is an inverse additive operator $(1-\lambda A)^{-1}$ with the domain K, which is given by the uniformly convergent series*

$$(1-\lambda A)^{-1} = \sum_{i=0}^{\infty} (\lambda A)^i. \tag{1}$$

The same holds if (12.1) *has a unique solution for any $y \in K$; however, the series* (16.1) *need not be uniformly convergent.*

The proof follows immediately from Definition 4, from Theorem 15, and from Alternative 6.

17. Example. As an example for Theorems 13 and 15, let us consider the additive operator A given by

$$A\,x(t) = x(a\,t), \qquad 0 < a \leqq 1.$$

Let its domain be the set K_p with $\alpha(t)=0$ and $\beta(t)=0$ (*viz*, 3.1); *i.e.*, the set of all non-negative functions belonging to $L_p(0, 1)$.

Let $x(t) \in K_p$ be the positive non-decreasing function determined by the equation $x(t) - x(at) = y(t)$. Since $y(t) \in K_p$, it follows from Theorem 13 that

$$x(t) = x_{ap}(t) + \lim_n x(a^n t).$$

From Theorem 15 and (6.3) we infer that the equation

$$x(t) - \lambda\,x(a\,t) = y(t), \qquad \lambda < a^{1/p},$$

for all $y \in K_p$, has a unique solution belonging to K_p. This solution is given by the uniformly convergent series x_{ap}.

18. Definition. An additive operator A having the domain K will be called uniform (*cf.* [2, 5]) if there is an integer n_2 and an element $s \in K$ such that

i) for all $x \in K$ the relation

$$\|x\|\,s - A^{n_2}\,x \in K \tag{1}$$

is satisfied; and

ii) for each $x \in K$, $x \neq 0$, there is a positive constant ε such that

$$A^{n_2}\,x - \varepsilon\,s \in K. \tag{2}$$

19. Theorem. *Equation* (12.1), *which is associated with a uniform operator A, has a solution for a $y_1 \in K$, $y_1 \neq 0$, if, and only if, $\lim_n \|A^n\| = 0$. This solution is unique.*

Proof. 1. It follows from Theorem 15 that the stated condition is sufficient.

2. In order to prove that the stated condition is necessary, let us assume that (12.1) has a solution for some $y_1 \in K$, $y_1 \neq 0$. Then it follows from Theorem 13 and (12.2) that $\lim_n \|A^n y_1\| = 0$. From (18.1), (18.2) and (2.2) we get, for any $y \in K$, the estimate

$$\|A^n y\| = \|A^{n-n_2} A^{n_2} y\| \leqq \|y\|\,\|A^{n-n_2} s\| \leqq \|y\|\,\|A^n y_1\|/\varepsilon_1. \tag{1}$$

Hence we have

$$\lim_n \|A^n\| \leqq \lim_n \|A^n y_1\|/\varepsilon_1 = 0. \qquad \text{Q.E.D.} \tag{2}$$

20. Lemma. *Let the spectral radius λ_0 of a uniform operator A be finite, and let $y \in K$, $y \neq 0$. Then $A^n y \neq 0$ for $n = 1, 2, \ldots$.*

Proof. Let $A^n y_1 = 0$ for $n = n'$ and $y_1 \in K$, $y_1 \neq 0$. Then it follows from (19.1) that $A^n = 0$ for all $n \geqq n'$, i.e., $\lambda_0 = \infty$, which proves the lemma.

21. Theorem. *Let the spectral radius λ_0 of a uniform operator A be finite, and let $y \in K$, $y \neq 0$. Then*

$$\lim_{\lambda < \lambda_0,\ \lambda \to \lambda_0} \|A^n (1 - \lambda A)^{-1} y\| = \infty, \qquad n = 0, 1, 2, \ldots,$$

and the approach to the limit is monotone.

Proof. The norm $\left\|\sum_{i=n}^{n+m} (\lambda_0 A)^i y\right\|$ and the integer m tend to infinity together for any $y \in K$, $y \neq 0$. For, in the opposite case it would follow from (1.4), from Lemma 12, and from Theorem 19, that $\lim_n \|(\lambda_0 A)^n\| = 0$, which would contradict the definition of λ_0. Thus we prove the theorem asserted, using Theorem 16 and the estimate

$$\left\|\sum_{i=n}^{\infty} (\lambda A)^i y\right\| \geqq \left\|\sum_{i=n}^{n+m} (\lambda/\lambda_0)^i (\lambda_0 A)^i y\right\| \geqq (\lambda/\lambda_0)^{n+m} \left\|\sum_{i=n}^{n+m} (\lambda_0 A)^i y\right\|, \qquad \lambda < \lambda_0.$$

Namely, here the right side is as great as desired, provided m is great enough and λ is close enough to λ_0, and, due to (2.2), the left side grows monotonely with λ.

22. Lemma. *Let the spectral radius λ_0 of a uniform operator A be finite. Then the elements*

$$\begin{aligned}\{x(n, \lambda)\colon\ x(n, \lambda) = \|A^n (1 - \lambda A)^{-1} y\|^{-1} A^n (1 - \lambda A)^{-1} y, \quad & y \in K, \\ y \neq 0, \quad n = 0, 1, 2, \ldots, \quad & \lambda \in (0, \lambda_0)\} \subset K\end{aligned} \tag{1}$$

satisfy the relations

$$\lim_{\lambda \to \lambda_0} \|(1 - \lambda_0 A)\, x(n, \lambda)\| = 0 \quad \textit{and} \quad \|x(n, \lambda)\| = 1. \tag{2}$$

Proof. The existence of elements $x(n, \lambda)$ follows from Theorem 16 and from Lemma 20. We prove the limit (22.2), using Theorem 21, as follows:

$$\lim_{\lambda \to \lambda_0} \|(1 - \lambda_0 A)\, x(n, \lambda)\| \leqq \|A^n\|\, \|y\| \lim_{\lambda \to \lambda_0} \|A^n (1 - \lambda A)^{-1} y\|^{-1} + \|A\| \lim_{\lambda \to \lambda_0} (\lambda_0 - \lambda) = 0.$$

23. Theorem. *Let the spectral radius λ_0 of a uniform operator A be finite. Then the homogeneous equation*

$$(1 - \lambda A)\, x = 0, \qquad x \in K, \qquad \|x\| = 1, \qquad \lambda \in (-\infty, \infty), \tag{1}$$

may have a solution only for $\lambda = \lambda_0$ (cf. [5]). If there is a solution x_0, it is unique and satisfies the relations

$$\lambda_0^{n_2} s - x_0 \in K \tag{2}$$

and

$$x_0 - \lambda_0^{n_2} \varepsilon_0 s \in K, \qquad \varepsilon_0 > 0. \tag{3}$$

Proof. 1. It follows from (23.1) and (2.1) that the eigenvalue λ must be positive. If λ_1 is an eigenvalue with an eigenvector x_1, then it follows from (23.1) that $\lim_n \|(\lambda_1 A)^n x_1\| = 1$, and as a consequence of Alternative 6 we get

$\lambda_1 \geqq \lambda_0$. But if λ_1 were greater than λ_0, then from the limit

$$\lim_n \|(\lambda_0 A)^n x_1\| = \lim_n (\lambda_0/\lambda_1)^n \|(\lambda_1 A)^n x_1\| = 0,$$

and from (19.2) we should get $\lim\limits_n \|(\lambda_0 A)^n\| = 0$, which contradicts the definition of λ_0. Consequently, the real eigenvalue can only be λ_0. Relations (23.2) and (23.3) follow immediately from (18.1), (18.2) and the fact that $\lambda_0 > 0$.

2. It remains to prove that $x_1 = x_2$ if x_1 and x_2 are the eigenvectors of the eigenvalue λ_0. To this purpose we need the supremum of all real constants a which satisfy the condition $x_1 - a x_2 \in K$. All such constants satisfy the relation $a \leqq \varepsilon_2^{-1}$, which follows from (23.2), (23.3) and (2.1). Since $a = 0$ satisfies the stated condition, the supremum exists, and, K being closed,

$$x_1 - (\sup a)\, x_2 \in K.$$

Applying the relation (18.2) to the difference $x_1 - (\sup a)\, x_2$, we get

$$x_1 - (\sup a)\, x_2 - \varepsilon s \in K.$$

From this relation and from (23.2) it follows that

$$x_1 - (\sup a + \varepsilon \lambda_0^{-n_2})\, x_2 \in K.$$

Comparing this relation with the definition of sup a, we infer that $\varepsilon = 0$, and as a consequence of Definition 18 we have $x_1 - (\sup a)\, x_2 = 0$; i.e., $x_1 = x_2$. Q.E.D.

24. Theorem. *Let the spectral radius λ_0 of a uniform operator A be finite, and let x_0 be its eigenvector. Then, for all $y \in K$, $0 \leqq \lambda < \lambda_0$, and $m \geqq 0$,*

$$\sum_{i=n_2+m}^{\infty} (\lambda A)^i y - \varepsilon_y \frac{(\lambda/\lambda_0)^{n_2+m}}{1-\lambda/\lambda_0}\, x_0 \in K \tag{1}$$

and

$$\frac{(\lambda/\lambda_0)^{n_2+m} \|y\|/\varepsilon_0}{1-\lambda/\lambda_0}\, x_0 - \sum_{i=n_2+m}^{\infty} (\lambda A)^i y \in K, \tag{2}$$

where ε_y equals the ε occurring in (18.2), with $x = y$.

Proof. Taking account of (23.2) and (23.3), we derive from the relations (18.1) and (18.2) the analogous ones

$$\|x\| \lambda_0^{-n_2} \varepsilon_0^{-1} x_0 - A^{n_2} x \in K \tag{3}$$

and

$$A^{n_2} x - \varepsilon \lambda_0^{-n_2} x_0 \in K. \tag{4}$$

Using them, we can immediately prove the theorem stated.

25. Theorem. *Let the spectral radius λ_0 of a uniform operator A be finite, let A be continuous* (i.e., *for all $x, y \in K$ we have $\|A x - A y\| \to 0$ if $\|x - y\| \to 0$*), *and let the n_3**th power of A be a compact operator. Then λ_0 is an eigenvalue* (cf. [5]), *and*

$$x_0 = \lim_{\lambda \to \lambda_0} x(n, \lambda) \in K, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

is the corresponding eigenvector, the norm of which equals 1.

Proof. First we prove the assertion for $n=n_3$, considering the infinite sequences

$$X_i=\{x(n_3,\lambda_{ik}):\ k=0,1,2,\dots;\ \lim_k \lambda_{ik}=\lambda_0\}.$$

Since the operator A is continuous and K is closed, it follows from (22.2) and from Theorem 23 that the limit $x_i=\lim\limits_k x(n_3,\lambda_{ik})$ of each convergent sequence X_i is the eigenvector x_0 belonging to the eigenvalue λ_0. The existence of one convergent sequence X_i, *i.e.*, the existence of x_0, follows from (22.1) and from the compactness of the operator A^{n_3}.

To complete the first part of the proof, we have to show that any sequence X_i converges. If a sequence X_i does not converge to x_0, it contains a subsequence X_j whose elements satisfy the relation $\|x(x_3,\lambda_{jk})-x_0\|>\delta>0$. But, due to the compactness of A^{n_3}, the sequence X_j has to contain a subsequence converging to x_0, which contradicts the definition of the sequence X_j. Hence any sequence X_i converges to x_0, proving the theorem for $n=n_3$.

Deriving the relation

$$\lim_{\lambda\to\lambda_0}\|x(n,\lambda)-x(n_3,\lambda)\|=0,\qquad n=0,1,2,\dots,$$

from Theorems 16 and 21, and from (22.1) and (1.3), we prove the limit (25.1) for all integer n.

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Lineare Operatoren mit positiver Inversen

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Vorgelegt von L. COLLATZ

Bei vielen Abschätzungen möchte man von Ordnungsbeziehungen zwischen bekannten „Ausgabewerten“ auf entsprechende Beziehungen zwischen den zugehörigen „Eingabewerten“ schließen. In welchen Fällen ist dies möglich? Abstrakt formuliert: Es sei M ein Operator mit halbgeordnetem Definitionsbereich und halbgeordnetem Wertebereich; wann folgt $u \leqq v$ aus $Mu \leqq Mv$? Einen Operator M mit dieser Eigenschaft nennt L. COLLATZ [4] „von monotoner Art“.

Bei Gleichungen der Form

$$Mu = r \tag{0.1}$$

lassen sich z.B. oft Schranken für eine Lösung u^* angeben, falls diese Aufgabe (0.1) von monotoner Art, d.h. falls der Operator M von monotoner Art ist. (Siehe L. COLLATZ [4] und [6], S. 42f.) Insbesondere erhält man dann Fehlerabschätzungen für eine Näherungslösung u_0. Dabei muß vorausgesetzt werden, daß u^* existiert. Mit Hilfe von Fixpunktsätzen kann man ferner in vielen Fällen die Existenz einer Lösung u^* von (0.1) beweisen und Schranken dafür herleiten, falls nur ein M benachbarter Operator M_0 von monotoner Art ist [16].

Wir betrachten hier im wesentlichen nur lineare Operatoren M. Der obigen Frage entspricht dann: Wann folgt $u \geqq o$ aus $Mu \geqq o$? oder: Wann besitzt M eine „positive“ Inverse? Die monotone Art nichtlinearer Operatoren läßt sich vielfach aus der monotonen Art linearer Operatoren folgern (s. Abschnitt 6.1). In einer anschließenden Arbeit sollen jedoch nichtlineare Operatoren direkt behandelt werden.

Probleme einer monotonen Art, welche sich nicht genau in der genannten Weise abstrakt beschreiben läßt, treten bei parabolischen Differentialgleichungen auf (s. z.B. [12], [18], [5], [13]). Auch bei Matrizen arbeitet man zum Teil mit anderen Relationen (z.B. [7]). Wir verallgemeinern daher die Fragestellung.

In einem archimedischen halbgeordneten linearen Raum führt die gegebene Ordnungsrelation $u \geqq o$ zu weiteren Beziehungen $u > o$, $u \succ o$ und $u \succeqq o$. Für endlichdimensionale Vektoren $u = (u^i)$ z.B. bedeutet bei der üblichen $\geqq$-Definition

$u > o$: alle u^i sind $\geqq 0$, aber nicht alle sind $= 0$,
$u \succ o$: alle u^i sind > 0.

Mit diesen verschiedenen Ordnungsbeziehungen definieren wir acht Klassen der monotonen Art linearer Operatoren. Wir zeigen, daß nur zwei davon „wesentlich verschieden“ sind. Eine dieser zwei Klassen enthält die oben genannte monotone

Art, die andere wird durch die stärkere Forderung „$u \succ o$ folgt aus $Mu > o$“ gekennzeichnet. Die letzte Eigenschaft findet man z.B. bei Randwertproblemen (Abschnitt 4). Die bekannte monotone Art bei parabolischen Differentialgleichungen dagegen gehört zur erstgenannten Klasse.

In Abschnitt 1 werden ferner hinreichende Kriterien für die monotone Art von Operatoren M bewiesen. Sie sind von der Form: M ist (in bestimmtem Sinne) von monotoner Art, wenn ein „größerer“ Operator A, d.h. ein Operator $A = M + B$ mit positivem B, (in bestimmtem, eventuell anderem Sinne) von monotoner Art ist und ein Element z mit $z \succ o$ und $Mz \geqq o$ (bzw. $Mz > o$ oder $Mz \succ o$) existiert. Die Beweise sind elementar und sehr einfach.

Praktisch hat man also eine monotone Art für einen größeren und *einfacheren* Operator A direkt nachzuweisen und ein geeignetes z zu konstruieren. Mit dieser Methode erhalten wir in den Abschnitten 2 bis 5 und 6.3 Bedingungen für die monotone Art bei verschiedenen Problemtypen. Bereits bekannte Resultate, zu deren Beweis man bisher zum Teil sehr verschiedenartige Mittel verwendete, werden so auf einheitliche und einfache Weise hergeleitet. Zum Beispiel kann man elliptische und parabolische Differentialgleichungen weitgehend gemeinsam behandeln. Darüber hinaus erhalten wir eine Reihe neuer Ergebnisse. Zum Beispiel werden teilweise allgemeinere Problemtypen behandelt, Voraussetzungen abgeschwächt, stärkere Behauptungen bewiesen. Bei Aufgaben mit parabolischen Differentialgleichungen wird gezeigt (wie bereits erwähnt), daß sie auch im Sinne der ursprünglichen Definition von L. Collatz [*4*] von monotoner Art sind.

Bei der Fehlerabschätzung für eine Aufgabe (0.1) nach der Methode von L. Collatz [*4*] muß man ein Element z der gewünschten Art ohnehin konstruieren. Praktisch bedeutet dies: Man braucht nur die monotone Art eines größeren Operators A, nicht die von M, zu beweisen; ist dann der Formalismus der Abschätzung durchführbar, sind die erhaltenen Schranken auch gültig (Satz 1.2). Hat man umgekehrt ein geeignetes z, läßt sich auch der Formalismus durchführen.

1. Ergebnisse für abstrakte Operatoren

1.1. $R = \{u, v, \ldots\}$ sei ein reeller halbgeordneter linearer Raum. Dieser Raum sei ferner als „archimedisch“ vorausgesetzt, d.h.:

$$\text{Aus } nu \leqq v \text{ für alle } n = 1, 2, \ldots \text{ folge } u \leqq o, \tag{1.1}$$

wenn u, v feste Elemente $\in R$ bedeuten und o das Nullelement ist.

Wir schreiben:

$u > o$, wenn $u \geqq o$ und $u \neq o$,
$u \succ o$, wenn zu jedem $v \in R$ eine natürliche Zahl n mit $nu > v$ existiert,
$u \succeq o$, wenn $u \succ o$ oder $u = o$.

Offenbar folgt $u > o$ aus $u \succ o$.

Es sei $z \in R$ ein Element $\succ o$ und v irgendein Element $\in R$, welches nicht $\prec o$ ist. Dann gibt es eine Zahl α mit

$$v \leqq \alpha z,$$

und für

$$\alpha_0 = \inf\{\alpha : -\infty < \alpha < \infty,\ v \leqq \alpha z\}$$

gilt
$$\alpha_0 \geqq 0. \tag{1.2}$$
Außerdem hat man
$$v \leqq \left(\alpha_0 + \frac{1}{n}\right) z, \quad \text{d.h.} \quad n(v - \alpha_0 z) \leqq z \qquad (n = 1, 2, 3, \ldots).$$
Aus der letzten Ungleichung folgt mit (1.1) $v - \alpha_0 z \leqq 0$, also
$$\alpha_0 = \min\{\alpha\colon -\infty < \alpha < \infty,\ v \leqq \alpha z\}. \tag{1.3}$$

Es seien nun R und S zwei solche archimedischen reellen halbgeordneten linearen Räume. Wir verwenden die gleichen Symbole $\geqq, >, \succ, \succeq$ für die Ordnungsrelationen in R und S. Selbst wenn R und S die gleichen Elemente haben, können diese Symbole jedoch in R und S etwas Verschiedenes bedeuten. Mit A, B, M bezeichnen wir im folgenden lineare Operatoren, die R in S abbilden.

Wir interessieren uns für Operatoren M, welche eine der folgenden Eigenschaften (a), …, (d), (a′), …, (d′) besitzen:

$$\begin{array}{llll} \text{(a)} & \text{aus } Mu > o \text{ folgt } u \succ o, & \text{(c)} & \text{aus } Mu > o \text{ folgt } u > o, \\ \text{(b)} & \text{aus } Mu \succ o \text{ folgt } u \succ o, & \text{(d)} & \text{aus } Mu \succ o \text{ folgt } u > o, \end{array} \tag{1.4}$$

$$\begin{array}{llll} \text{(a}'\text{)} & \text{aus } Mu \geqq o \text{ folgt } u \succeq o, & \text{(c}'\text{)} & \text{aus } Mu \geqq o \text{ folgt } u \geqq o, \\ \text{(b}'\text{)} & \text{aus } Mu \succeq o \text{ folgt } u \succeq o, & \text{(d}'\text{)} & \text{aus } Mu \succeq o \text{ folgt } u \geqq o. \end{array} \tag{1.5}$$

L. COLLATZ [*4*] nannte M einen „Operator monotoner Art“, falls M die Eigenschaft (c′) besitzt. Wir betrachten hier also nun verschiedene Sorten von „Operatoren monotoner Art“. Eine etwa (b) entsprechende Eigenschaft wurde bereits bei parabolischen Differentialgleichungen benutzt ([*12*], [*18*], [*5*], [*13*]). Die folgenden zwei Hilfssätze erläutern die Beziehungen zwischen den verschiedenen Eigenschaften in (1.4) und (1.5).

Hilfssatz 1.1. 1) *Jede der Eigenschaften in* (1.4) *folgt aus der entsprechenden Eigenschaft in* (1.5). *Wenn* M^{-1} *existiert, sind die Eigenschaften in* (1.4) *und* (1.5) *paarweise äquivalent.*

2) *Wenn* M *eine der Eigenschaften* (1.5) *besitzt, existiert* M^{-1}.

3) M *habe eine der Eigenschaften* (1.4), *und es gebe ein Element* $z \in R$, *derart daß*
$$\begin{array}{l} Mz > o \text{ im Falle (a) oder (c)}, \\ Mz \succ o \text{ im Falle (b) oder (d)}. \end{array} \tag{1.6}$$
Dann existiert M^{-1}.

Beweis. 1) ist klar. Zum Beweis von 2) und 3) sei $\varphi \in R$ mit $M\varphi = o$. Es ist jeweils $\varphi = o$ zu zeigen.

2) Es gilt auch $M(-\varphi) = o$. Mit jeder der Eigenschaften (1.5) folgt daher sowohl $\varphi \geqq o$ als auch $-\varphi \geqq o$, d.h. $\varphi = o$.

3) Die Elemente $z + n\varphi$ und $z - n\varphi$ $(n = 1, 2, \ldots)$ genügen derselben Ungleichung wie z in (1.6). In jedem der Fälle folgt daraus
$$z + n\varphi > o \quad \text{und} \quad z - n\varphi > o \qquad (n = 1, 2, \ldots),$$
d.h.
$$n(-\varphi) < z \quad \text{und} \quad n\varphi < z \qquad (n = 1, 2, \ldots).$$

Da R archimedisch ist, erhält man hiermit wiederum $-\varphi \leqq o$ und $\varphi \leqq o$, d.h. $\varphi = o$.

In praktischer Hinsicht ist es keine wesentliche Einschränkung, zu fordern, daß ein Element z mit (1.6) existiert. Dies bedeutet nur, daß es überhaupt einen Sinn hat, mit der entsprechenden Eigenschaft in (1.4) zu arbeiten. Wir werden im folgenden bei der theoretischen Untersuchung hauptsächlich die Fälle (1.4) betrachten.

Mit den verschiedenen Ordnungsdefinitionen kann man auch verschiedene Arten der Positivität von linearen Operatoren C definieren, welche eine Teilmenge $D_C < S$ in R abbilden:

$$\begin{aligned}
&(\text{a}'') \text{ aus } v > o \ (v \in D_C) \text{ folgt } Cv \succ o,\\
&(\text{b}'') \text{ aus } v \succ o \ (v \in D_C) \text{ folgt } Cv \succ o,\\
&(\text{c}'') \text{ aus } v > o \ (v \in D_C) \text{ folgt } Cv > o,\\
&(\text{d}'') \text{ aus } v \succ o \ (v \in D_C) \text{ folgt } Cv > o
\end{aligned} \tag{1.7}$$

und entsprechende Eigenschaften mit dem Gleichheitszeichen.

Falls M^{-1} existiert, hat M genau dann eine der Eigenschaften (1.4), wenn $C = M^{-1}$ die entsprechende Eigenschaft in (1.7) besitzt.

Der folgende Hilfssatz gibt Beziehungen zwischen den verschiedenen Eigenschaften in (1.4) an.

Hilfssatz 1.2. 1) *Aus* (a) *folgen* (b), (c) *und* (d). *Sowohl* (b) *als auch* (c) *haben* (d) *zur Folge.*

2) *Wenn es ein* $z_1 \in R$ *mit*

$$M z_1 \succ o \tag{1.8}$$

gibt, folgt (c) *aus* (d).

3) *Wenn es ein* $z_2 \in R$ *mit*

$$z_2 \succ o \tag{1.9}$$

gibt, folgt (b) *aus* (d).

4) *Wenn es sowohl ein* $z_1 \in R$ *mit* (1.8) *als auch ein* $z_2 \in R$ *mit* (1.9) *gibt, sind die Eigenschaften* (b), (c) *und* (d) *äquivalent.*

Beweis. 1) ist klar.

2) M habe die Eigenschaft (d), und es sei $Mu > o$. Dann gilt:

$$M(nu + z_1) = nMu + Mz_1 \succ o \qquad (n = 1, 2, 3, \ldots)$$

und daher wegen (d)

$$nu + z_1 > o, \quad \text{d.h.} \quad n(-u) < z_1 \qquad (n = 1, 2, 3, \ldots).$$

Daraus folgt $(-u) < o$, d.h. $u > o$ ($Mu > o$ schließt $u = o$ aus).

3) M habe wiederum die Eigenschaft (d), es sei aber nun $Mu \succ o$. Dann ist

$$M(nu - z_2) = nMu - Mz_2 \succ o$$

für ein genügend großes n und daher wegen (d)

$$nu - z_2 > o, \quad \text{d.h.} \quad u > \frac{1}{n} z_2 \succ o.$$

4) folgt unmittelbar aus 2) und 3).

Die Bedingung, daß Elemente z_1 und z_2 mit (1.8) bzw. (1.9) existieren, ist wiederum keine wesentliche Einschränkung in praktischer Sicht. Der folgende Satz gibt eine Antwort auf die Frage: Unter welchen Bedingungen besitzt $M = A - B$ eine der Eigenschaften in (1.4) oder (1.5), wenn A eine dieser Eigenschaften hat?

Satz 1.1. *A und B seien lineare Operatoren, die R in S abbilden, und folgende Voraussetzungen seien erfüllt:*

i) *aus $u \geqq o$ folgt $Bu \geqq o$ $(u \in R)$,* (1.10)

ii) *es gibt ein $z \in R$ mit*

$$z \succ o \quad \text{und} \quad (A - B)\, z \geqq o. \tag{1.11}$$

Dann gilt:

1) *Wenn A die Eigenschaft* (a) *besitzt, hat $A - B$ die Eigenschaften* (a), (b), (c) *und* (d).

2) *Wenn A die Eigenschaft* (b) *besitzt, hat $A - B$ die Eigenschaften* (b) *und* (d).

3) *Ist*

$$(A - B)\, z > o \tag{1.12}$$

und besitzt A eine der Eigenschaften (a) *oder* (a′), *so hat $A - B$ alle Eigenschaften in* (1.4) *und* (1.5) *und $(A - B)^{-1}$ existiert.*

4) *Ist*

$$(A - B)\, z \succ o \tag{1.13}$$

und besitzt A eine der Eigenschaften (b), (c), (d), (b′), (c′) *oder* (d′), *so hat $A - B$ alle diese Eigenschaften und $(A - B)^{-1}$ existiert.*

Beweis. A habe die Eigenschaft (a) oder die Eigenschaft (b) (Fall (a) bzw. (b)), und es sei u ein Element $\in R$ mit

$$(A - B)\, u \begin{cases} > o & \text{im Falle (a),} \\ \succ o & \text{im Falle (b).} \end{cases} \tag{1.14}$$

Wir wollen in beiden Fällen

$$u \succ o \tag{1.15}$$

beweisen und nehmen an, diese Ungleichung sei nicht richtig. Dann ist $v = -u$ nicht $\prec o$, und man hat (1.2) für die durch (1.3) gegebene Konstante α_0. Aus (1.14), und (1.11) folgt

$$A\, v \left\{ \begin{matrix} < \\ \prec \end{matrix} \right\} B\, v \leqq \alpha_0 B\, z \leqq \alpha_0 A\, z \begin{cases} \text{im Falle (a),} \\ \text{im Falle (b),} \end{cases}$$

also

$$A(\alpha_0 z - v) \begin{cases} > o & \text{im Falle (a),} \\ \succ o & \text{im Falle (b).} \end{cases}$$

In beiden Fällen ergibt sich daraus

$$\alpha_0 z - v \succ o.$$

Dies hat zur Folge, daß

$$n(\alpha_0 z - v) > z, \quad \text{d.h.} \quad v < \left(\alpha_0 - \frac{1}{n}\right) z$$

für ein genügend großes n, im Widerspruch zur Definition von α_0. Deshalb gilt (1.15).

Dies beweist, daß $A-B$ die Eigenschaft (a) bzw. (b) hat, wenn A diese Eigenschaft besitzt. Die übrigen Behauptungen folgen mit den Hilfssätzen 1.1 und 1.2 (auf $A-B$ bzw. A angewendet).

1.2. Es sei nun eine Gleichung

$$(A-B)\,u=s \tag{1.16}$$

gegeben. A und B bilden R in S ab, s sei ein festes Element $\in S$. Für diese Gleichung sei eine Näherungslösung u_0 bekannt;

$$d[u_0]=-(A-B)\,u_0+s \tag{1.17}$$

bezeichnet ihren Defekt. Mit Satz 1 erhält man für dieses Problem eine Fehlerabschätzung:

Satz 1.2. i) *Der Operator A sei von der Art* (a) *oder* (b) *in* (1.4) (*Fall* (a) *bzw.* (b)).

ii) *Für den Operator B gelte* (1.10).

iii) *Es sei $s \subset (A-B)R$, d.h. die Gleichung* (1.16) *besitze eine Lösung $u^* \in R$.*

iv) *Es gebe Elemente $v, w \in R$, derart daß*

$$v \prec w \tag{1.18}$$

und

$$\left.\begin{aligned} &(A-B)\,v \leqq d[u_0] \leqq (A-B)\,w \quad \textit{im Falle } \text{(a)},\\ &(A-B)\,v \preceqq d[u_0] \preceqq (A-B)\,w \quad \textit{im Falle } \text{(b)}. \end{aligned}\right\} \tag{1.19}$$

Unter diesen Voraussetzungen gilt: Entweder sind beide Funktionen u_0+v und u_0+w Lösungen von (1.16), *oder* (1.16) *besitzt nur die Lösung u^*. Im letzten Falle ist*

$$v \preceqq u^*-u_0 \preceqq w, \tag{1.20}$$

wobei links bzw. rechts das Gleichheitszeichen genau dann richtig ist, wenn in (1.19) *das Gleichheitszeichen auf der entsprechenden Seite gilt.*

Beweis. Es sind genau dann beide Funktionen u_0+v und u_0+w Lösungen von (1.16), wenn in (1.19) auf beiden Seiten das Gleichheitszeichen gilt. Es werde nun angenommen, daß dieser Fall nicht vorliegt. Dann hat man

$$\begin{aligned} &(A-B)\,z > o \quad \text{im Falle (a)},\\ &(A-B)\,z \succ o \quad \text{im Falle (b)} \end{aligned}$$

mit

$$z=w-v \succ o\,.$$

Nach Satz 1.1 existiert daher $(A-B)^{-1}$, und die Lösung u^* ist eindeutig.

Der Defekt kann in der Form

$$d[u_0]=(A-B)\,(u^*-u_0) \tag{1.21}$$

geschrieben werden. Wir betrachten jeweils die rechte Ungleichung in (1.19). Wenn dort das Gleichheitszeichen richtig ist, hat man $-(A-B)\,(u_0+w)+s=o$, und die eindeutige Lösung ist daher $u^* = u_0 + w$, so daß in (1.20) $u^* - u_0 = w$ gilt. Im anderen Falle ($<$ bzw. $\prec$) ergibt Satz 1.1 die Aussage $u^* - u_0 \prec w$, wenn man (1.21) beachtet.

Die linke Ungleichung in (1.19) kann in derselben Weise behandelt werden.

Bei Anwendung des Satzes 2 konstruiert man also mit v und w auch ein Element $z = w - v$, wie es in Satz 1.1 benötigt wird. Kennt man andererseits ein $z \in R$ mit

$$z \succ o \quad \text{und} \quad (A - B)\, z \succ o,$$

so genügen die Elemente

$$v = \alpha z, \quad w = \beta z$$

der Voraussetzung (1.19), wenn man die Konstanten α und β geeignet wählt.

Anwendungen

Die abstrakten Ergebnisse lassen sich in mannigfacher Weise auf konkrete Probleme anwenden. Die folgenden Abschnitte geben einige Beispiele. In diesen Beispielen sind die Elemente der Räume R und S Vektoren mit skalaren Komponenten, Funktionen, Vektoren mit Funktionen als Komponenten oder Kompositionen solcher Größen. Die Definition $u \geqq o$ ist dabei — falls nichts anderes bemerkt wird — immer punkt- und koordinatenweise zu verstehen. Dies bezeichnen wir als die *übliche* Ordnungsdefinition. Es habe $u \in R$ (oder S) z.B. die Form $u = \langle \varphi(t), \omega \rangle$ mit einem Vektor $\varphi(t) = \big(\varphi^k(t)\big)_{k=1,\ldots,n}$ und einem Vektor $\omega = (\omega^i)_{i=1,2,\ldots,m}$. Die $\varphi^k(t)$ seien auf einer Menge G erklärte Funktionen, die ω^i skalar. Dann bedeutet $u \geqq o$ im Sinne der üblichen Ordnungsdefinition[1]:

$$\varphi^k(t) \geqq 0 \quad (k = 1, 2, \ldots, n;\ t \in G), \qquad \omega^i \geqq 0 \quad (i = 1, 2, \ldots, m).$$

Alle vorkommenden Größen werden als reell vorausgesetzt.

2. Matrizen

Die Elemente von R und S seien n-dimensionale Vektoren $u = (u^i)$ mit (reellen) Komponenten u^i $(i = 1, 2, \ldots, n)$. Die linearen Operatoren von R in S und ebenso die von S in R können dann mit den $n \times n$-Matrizen $A = (a_{ij})$, $B = (b_{ij})$, … identifiziert werden.

Bei der üblichen Ordnungsdefinition in R und S bedeutet:

$$u \succ o \quad \text{für} \quad u \in R \quad \text{oder } S\colon \quad u^i > o \qquad (i = 1, 2, \ldots, n). \tag{2.1}$$

Eine $n \times n$-Matrix C hat eine der Eigenschaften in (1.7) bei $D_C = S$, wenn

$$c_{ij} \geqq 0 \qquad (i, j = 1, 2, \ldots, n)$$

ist und außerdem beziehungsweise gilt:

(a″) alle c_{ij} sind > 0,
(b″) es gibt in jeder Zeile mindestens ein $c_{ij} > 0$,
(c″) es gibt in jeder Spalte mindestens ein $c_{ij} > 0$,
(d″) es gibt mindestens ein $c_{ij} > 0$.

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[1] Sind in einer Ungleichung die Argumente einer Funktion hingeschrieben, so ist diese Ungleichung punktweise zu verstehen. Sind die Argumente weggelassen, ist die Funktion als Element des betrachteten halbgeordneten Raumes anzusehen und die Ungleichung als Ordnungsbeziehung in diesem Raum aufzufassen.

Hier sieht man daher leicht unmittelbar, daß die Eigenschaften (b''), (c'') und (d'') äquivalent sind, falls $C^{-1}=M$ existiert. In diesem Falle sind jedoch auch die entsprechenden Eigenschaften für M in (1.4) und für $C=M^{-1}$ in (1.7) äquivalent. Wegen dieser Beziehungen ergibt Satz 1.1:

Satz 2.1. *A und B seien zwei $n\times n$-Matrizen mit folgenden Eigenschaften:*

i) $A^{-1}=(\alpha_{ij})$ *existiert,*

ii) *es ist entweder* $\alpha_{ij}>0 \quad (i,j=1,2,\ldots,n) \quad$ (*Fall* (a)),
oder $\alpha_{ij}\geqq 0 \quad (i,j=1,2,\ldots,n) \quad$ (*Fall* (b)),

iii) $$b_{ij}\geqq 0 \quad (i,j=1,2,\ldots,n), \tag{2.2}$$

iv) *es gibt einen Vektor* z *mit*

$$z\succ o$$

und

$$\begin{aligned}(A-B)\,z &> o \quad \textit{im Falle (a)},\\ (A-B)\,z &\succ o \quad \textit{im Falle (b)}\end{aligned} \tag{2.3}$$

(*im Sinne von* (2.1)).

Dann existiert $(A-B)^{-1}=(\mu_{ij})$, *und es gilt*

$$\mu_{ij}>0 \quad (i,j=1,2,\ldots,n) \quad \textit{im Falle (a)}, \tag{2.4}$$

$$\mu_{ij}\geqq 0 \quad (i,j=1,2,\ldots,n) \quad \textit{im Falle (b)}. \tag{2.5}$$

Im Spezialfall einer Diagonalmatrix A und einer nicht zerlegbaren Matrix B läßt sich mehr aussagen, als in Satz 2.1, Fall (b) enthalten ist.

Satz 2.2. *A und B seien $n\times n$-Matrizen mit folgenden Eigenschaften:*

i) *A ist eine Diagonalmatrix mit Diagonalelementen* $a_{ii}>o$.

ii) *B ist nicht zerlegbar und genügt* (2.2),

iii) *es gibt einen Vektor* $z\succ o$ *mit*

$$(A-B)\,z>o. \tag{2.6}$$

Dann existiert $(A-B)^{-1}=(\mu_{ij})$ *und es gilt* (2.4).

Beweis. A und B genügen den Voraussetzungen i), ii) und iii) des Satzes 2.1, Fall (b). Mit z beginnend kann man einen Vektor $z'\succ o$ konstruieren, derart daß $(A-B)\,z'\succ o$ gilt. (Dies Verfahren beschreibt HOUSEHOLDER in [9], S. 319.) Satz 2.1 sagt daher aus, daß $(A-B)^{-1}$ existiert und (2.5) gilt. (2.5) bedeutet: aus

$$(A-B)\,u>o \tag{2.7}$$

folgt $u>o$. Wir zeigen nun, daß (2.7) sogar $u\succ o$ zur Folge hat, welche Eigenschaft (2.4) äquivalent ist.

Es werde angenommen, daß $u>o$, aber nicht $u\succ o$ gilt. Dann existiert eine Nummer l mit

$$1\leqq l<n$$

und

$$\begin{aligned}u^i &= 0 \quad \text{für} \quad i=i_1,i_2,\ldots,i_l,\\ u^i &> 0 \quad \text{für} \quad i=i_{l+1},\ldots,i_n.\end{aligned}$$

Es gibt mindestens ein Element $b_{i_p i_q}>0$ mit $1\leqq p\leqq l$, $l+1\leqq q\leqq n$, da B nicht zerlegbar ist. Daraus folgt, daß die i_p-te Komponente von $(A-B)\,u$ im Widerspruch zu (2.7) <0 ist.

Beispiel. Es sei $M=(m_{ij})$ eine $n\times n$-Matrix mit

$$m_{ij}=\begin{cases} c_i & \text{für } i=j, \\ -1 & \text{für } i=j+1 \text{ und } i=j-1, \\ 0 & \text{sonst,} \end{cases}$$

wobei die c_i Konstanten bedeuten, derart daß

$$2\cos\frac{\pi}{n+1}\begin{cases} \leqq c_i & \text{für alle } i \\ < c_i & \text{für mindestens ein } i \end{cases} \qquad (i=1,2,\ldots,n).$$

M kann in der Form

$$M=A-B \quad \text{mit} \quad A=(c_i\,\delta_{ij})$$

geschrieben werden. A und B genügen den Voraussetzungen i) und ii) des Satzes 2.2, und die Voraussetzung iii) ist mit

$$z=(z^i), \quad z^i=\sin\frac{i\,\pi}{n+1} \qquad (i=1,2,\ldots,n)$$

ebenfalls erfüllt wegen

$$\begin{aligned}(A-B)\,z=(2I-B)\,z+(A-2I)\,z &= 2\left(1-\cos\frac{\pi}{n+1}\right)z+\left((c_i-2)\,z^i\right)\\ &=\left(\left(c_i-2\cos\frac{\pi}{n+1}\right)z^i\right).\end{aligned}$$

Deshalb existiert M^{-1} und hat nur Elemente >0.

G. SCHULZ [*17*] zeigte mit geometrischen Überlegungen, daß (2.5) gilt, wenn A eine Diagonalmatrix mit Diagonalelementen >0 ist, $b_{ij}>0$ $(i,j=1,2,\ldots,n)$ statt (2.2) gilt, Voraussetzung iv), Fall (b), von Satz 2.1 erfüllt ist und $(A-B)^{-1}$ existiert. In [*4*] leitete L. COLLATZ das Ergebnis des Satzes 2.1, Fall (b), her mit der Einschränkung, daß A diagonal und (2.3) mit $z^i=1$ $(i=1,2,\ldots,n)$ erfüllt sein soll. Ferner bewies COLLATZ in [*4*] den Satz 2.2 mit der Einschränkung, daß (2.6) wieder mit $z^i=1$ $(i=1,2,\ldots,n)$ gefordert und (2.5) statt (2.4) behauptet wird. Das Ergebnis des Satzes 2.1, Fall (b), für $A=I$ findet man bei KY FAN [*7*]. Die allgemeine Behauptung des Falles (b) läßt sich leicht aus diesem Spezialfall herleiten. Beziehungen zu diesem Satz, Fall (b), haben auch determinantentheoretische Arbeiten von A. OSTROWSKI [*14*] und H. SCHNEIDER [*15*]. L. COLLATZ zeigte in [*6*] (S. 44f.), daß Satz 2.1, Fall (b), für den Fall einer Diagonalmatrix A richtig ist, und er bewies ferner Satz 2.2 mit der Einschränkung, daß (2.5) statt (2.4) behauptet wird.

Man kann auch allgemeinere Ordnungsdefinitionen benutzen. Es sei z.B. $H=(h_{ij})$ eine nicht singuläre Matrix, und

$$u\underset{H}{\geqq}o \quad \text{bedeute} \quad \sum_{j=1}^{n}h_{ij}\,u^j\geqq 0 \qquad (i=1,2,\ldots,n).$$

Mit dieser Ordnungsdefinition in R und einer entsprechenden Definition $u\underset{K}{\geqq}o$ in S lassen sich entsprechende Resultate herleiten. Man hat z.B. die Matrix $K A H^{-1}$ an Stelle von A zu betrachten. Es sei — ohne Beweis — nur ein einfaches Beispiel genannt:

Beispiel. Die Matrix $F=(f_{ij})$ mit

$$f_{ij}=\begin{cases} c_i & \text{für } i=j, \\ 1 & \text{für } i=j+1 \text{ und } i=j-1, \\ 0 & \text{sonst} \end{cases}$$

besitzt eine Inverse $F^{-1}=(\varphi_{ij})$, welche die Eigenschaft (a″) bezüglich der Ordnungsdefinition $u \underset{H}{\geqq} o$ in R und S mit $H=\left((-1)^{i+j}\,\delta_{ij}\right)$ hat. Das bedeutet

$$(-1)^{i+j}\,\varphi_{ij}>0 \qquad (i,j=1,2,\dots,n).$$

3. Anfangswertaufgaben bei gewöhnlichen Differentialgleichungen erster Ordnung

Wir betrachten (formale) Differentialoperatoren der Form

$$L_i[u]=\frac{du^i}{dt}-\sum_{j=1}^{n} a_{ij}(t)\,u^j(t) \qquad (i=1,2,\dots,n)$$

für n Funktionen $u^j(t)$ im Intervall $[0,T]$. Diese Funktionen werden zu einem Vektor $u(t)=\left(u^j(t)\right)$ zusammengefaßt. Die $a_{ij}(t)$ seien stetig in $[0,T]$, und es gelte

$$a_{ij}(t)\geqq 0 \quad \text{für} \quad i\neq j \quad (i,j=1,2,\dots,n;\ 0\leqq t\leqq T). \tag{3.1}$$

Im folgenden wird $u(t)$ immer als stetig differenzierbar vorausgesetzt. Der abstrakte Satz 1.1 läßt sich in verschiedener Weise anwenden. Wir beweisen als Beispiel:

Satz 3.1. *Unter der Voraussetzung* (3.1) *gilt beziehungsweise: Aus*

$$L_i[u](t)\begin{cases} >0 \\ \geqq 0 \\ >0 \\ \geqq 0 \end{cases} (0\leqq t\leqq T) \quad \textit{und} \quad u^i(0)\begin{cases} =0 \\ =0 \\ >0 \\ \geqq 0 \end{cases} \quad (i=1,2,\dots,n)$$

folgt (3.2)

$$\begin{cases} u^i(t)>0 & (0<t\leqq T) \quad \textit{und} \quad \dot{u}^i(0)>0 \\ u^i(t)\geqq 0 & (0\leqq t\leqq T) \\ u^i(t)>0 & (0\leqq t\leqq T) \\ u^i(t)\geqq 0 & (0\leqq t\leqq T) \end{cases} \qquad (i=1,2,\dots,n).$$

Beweis. Mit $C_n[0,T]$ bezeichnen wir die Menge der Vektoren $\varphi=\left(\varphi^i(t)\right)$ mit n auf $[0,T]$ stetigen Komponenten $\varphi^i(t)$.

1) Zum Beweis der Behauptungen in den ersten beiden Zeilen von (3.2) setzen wir

$$R=\left\{u\in C_n[0,T]:\ \ \dot{u}^i(t) \text{ stetig},\ \ u^i(0)=0 \ \ (i=1,2,\dots,n)\right\}, \qquad S=C_n[0,T]$$

und definieren A und B auf R durch[2]

$$A\,u=(\dot{u}^i-a_{ii}\,u^i), \qquad B\,u=\Bigl(\sum_{j=1}^{n}{}'\,a_{ij}\,u^j\Bigr).$$

---

[2] Der Strich an der Summe bedeutet: es wird nur über $j\neq i$ summiert.

Bei der üblichen Ordnungsdefinition bedeutet

$u \succ o$ für $u \in R$: $u^i(t) > 0$ $(0 < t \leqq T)$ und $\dot{u}^i(0) > 0$ $(i = 1, 2, \ldots, n)$,

$v \succ o$ für $v \in S$: $v^i(t) > 0$ $(0 \leqq t \leqq T)$.

Der Operator A hat die Eigenschaft (b), wie man durch explizite Lösung der Gleichung $Au = s$ erkennt. Offenbar genügt ferner B der Ungleichung (1.10). Für das Element

$$z = (z^i) \in R \quad \text{mit} \quad z^i(t) = t\, e^{Nt}$$

und

$$\sum_{j-1}^{n} a_{ij}(t) < N \qquad (0 \leqq t \leqq T,\ i = 1, 2, \ldots, n) \tag{3.3}$$

gilt schließlich $z \succ o$ und (1.13). Daher besitzt $A - B$ nach Satz 1.1 die Eigenschaften (b) und (c′). Das ist die Behauptung der ersten beiden Zeilen in (3.2).

2) Zum Beweis der restlichen Behauptungen definieren wir:

$$R = \{u \in C_n[0, T]:\ \dot{u}^i(t) \ \text{stetig}\ (i = 1, 2, \ldots, n)\},$$

S ist die Menge der Kompositionen $v = \langle \varphi, \omega \rangle$ mit $\varphi \in C_n[0, T]$ und einem reellen n-dimensionalen Vektor $\omega = (\omega^i)$,

$$A u = \langle (\dot{u}^i - a_{ii} u^i),\ (u^i(t_0)) \rangle,$$

$$B u = \left\langle \left(\sum_{j=1}^{n}{}' a_{ij} u^j \right), o \right\rangle.$$

Bei der üblichen Ordnungsdefinition in R und S bedeutet

$u \succ o$ für $u \in R$: $u^i(t) > 0$ $(0 \leqq t \leqq T;\ i = 1, 2, 3, \ldots, n)$,

$v \succ o$ für $v = \langle \varphi, \omega \rangle \in S$: $v^i(t) > 0$ $(0 \leqq t \leqq T)$ und $\omega^i > 0$ $(i = 1, 2, \ldots, n)$.

Wieder erkennt man durch explizite Lösung der Aufgabe $Au = s$, daß A die Eigenschaft (b) hat. Für B gilt (1.10), und

$$z = (z^i) \quad \text{mit} \quad z^i(t) = e^{Nt} \quad \text{und (3.3)}$$

erfüllt (1.11) und (1.13). Die Eigenschaften (b) und (c′) von $A - B$ sind in diesem Falle mit den letzten beiden Behauptungen in (3.2) identisch.

Für den Fall konstanter a_{ij} $(i, j = 1, 2, \ldots, n)$ folgen die in der zweiten und vierten Zeile von (3.2) enthaltenen Ergebnisse aus einem Satz bei R. BELLMAN [*3*] (S. 172) über die Positivität der Matrix e^{At}.

Mit Satz 3.1 lassen sich auch leicht Aussagen für Differentialgleichungen höherer Ordnung herleiten, indem man diese in der üblichen Weise auf Differentialgleichungen erster Ordnung zurückführt. Zum Beispiel kann man so ein Ergebnis von L. COLLATZ [*4*] (S. 372) unter schwächeren Voraussetzungen beweisen.

4. Randwertaufgaben bei gewöhnlichen Differentialgleichungen zweiter Ordnung

4.1. Es sei

$$L[u] = L[u](t) = -p_2(t)\, \ddot{u} - p_1(t)\, \dot{u} + p_0(t)\, u$$

ein im Intervall $0 \leqq t \leqq 1$ gegebener Differentialausdruck für eine Funktion $u(t)$ mit stetigen $p_i(t)$ $(i=0, 1, 2)$ und

$$p_2(t) > 0 \qquad (0 \leqq t \leqq 1).$$

Daneben betrachten wir die Randausdrücke[3]

$$U[u] = \alpha\, u(0) - \beta\, \dot{u}(0), \qquad V[u] = \gamma\, u(1) + \delta\, \dot{u}(1).$$

α, β, γ und δ seien gegebene Konstanten mit

$$\beta \geqq 0, \qquad \delta \geqq 0.$$

Alle vorkommenden Funktionen $u(t)$ werden in diesem Abschnitt 4 als zweimal stetig differenzierbar auf $[0, 1]$ vorausgesetzt.

4.2. Es werden eine stetige Funktion $P_0(t)$ und Konstanten a, c gewählt, derart daß

$$P_0(t) \geqq p_0(t), \qquad P_0(t) > 0 \qquad (0 \leqq t \leqq 1),$$

$$a \geqq \max(\alpha, 0), \qquad c \geqq \max(\gamma, 0), \qquad a + \beta > 0, \qquad c + \delta > 0.$$

Wir definieren dann

$$L_A[u] = -\, p_2\, u - p_1\, u + P_0\, u,$$

$$U_A[u] = a\, u(0) - \beta\, u(0), \qquad V_A[u] = c\, u(1) - \delta\, u(1).$$

Für den Augenblick sei nun $u(t)$ eine fest gewählte (zweimal stetig differenzierbare) Funktion, derart daß

$$L_A[u](t) \geqq 0 \qquad (0 \leqq t \leqq 1) \tag{4.1}$$

gilt. In einem Punkte $t_0 \in [0, 1]$ nehme $u(t)$ ein relatives Minimum an.

Ist $0 < t_0 < 1$, so kann nicht

$$u(t_0) < 0 \tag{4.2}$$

sein, da diese Ungleichung zusammen mit $\dot{u}(t_0) = 0$ und $\ddot{u}(t_0) \geqq 0$ der Voraussetzung (4.1) widerspricht.

Gilt neben (4.1)

$$U_A[u] \geqq 0, \tag{4.3}$$

so kann (4.2) auch im Falle $t_0 = 0$ nicht richtig sein. (4.2) schließt den Fall $\beta = 0$ aus. Im Falle $\beta > 0$, $a > 0$ ergeben (4.2) und (4.3) $\dot{u}(0) < 0$, so daß dann kleinere Werte als $u(0)$ in jeder (rechtsseitigen) Umgebung von $t_0 = 0$ existieren. Für $\beta > 0$, $a = 0$ kann man nur auf $\dot{u}(0) = 0$ schließen, jedoch ergibt (4.1) dann $\ddot{u}(0) < 0$, und $u(t)$ kann daher in $t = 0$ ebenfalls kein relatives Minimum besitzen.

Ebenso zeigt man, daß (4.1) und $V_A[u] \geqq 0$ den Fall $t_0 = 1$, $u(1) < 0$ ausschließen.

4.3. Es sei nun zunächst

$$\alpha \geqq 0, \qquad \gamma \geqq 0, \qquad \alpha + \beta > 0, \qquad \gamma + \delta > 0 \tag{4.4}$$

und

$$a = \alpha, \qquad c = \gamma$$

---

[3] Sind die $p_i(t)$ periodisch (mit der Periode 1), lassen sich in ähnlicher Weise auch Ergebnisse für die Randausdrücke $U[u] = u(0) - u(1)$, $V[u] = \dot{u}(0) - \dot{u}(1)$ herleiten.

vorausgesetzt. Wir definieren dann

$$R = \{u \in C[0,1] \colon \ddot{u}(t) \text{ stetig, } U[u] = V[u] = 0\},$$
$$S = C[0,1]$$

und verwenden in R und S die übliche Ordnungsdefinition. Dann bedeutet

$$u \succ o \quad \text{für} \quad u \in R\colon \begin{cases} u(t) > 0 & (0 < t < 1) \\ u(0) > 0 \text{ für } \beta \neq 0, & \dot{u}(0) > 0 \text{ für } \beta = 0, \\ u(1) > 0 \text{ für } \delta \neq 0, & -\dot{u}(1) > 0 \text{ für } \delta = 0, \end{cases} \tag{4.5}$$

$$v \succ o \quad \text{für} \quad v \in S\colon \qquad v(t) > 0 \qquad (0 \leqq t \leqq 1).$$

A und B seien auf R durch

$$A\,u = L_A[u], \qquad B\,u = \big(P_0(t) - p_0(t)\big)\,u$$

erklärt, so daß

$$(A - B)\,u = L[u].$$

Wie die Überlegungen unter 4.2 zeigen, folgt aus

$$A\,u \geqq o \qquad (\text{für } u \in R),$$

daß $u \geqq o$ gilt. Das heißt der Operator A hat die Eigenschaft (c′).

4.4. Zu zeigen, daß A auch die Eigenschaft (a) besitzt, erfordert zusätzliche Überlegungen. Es seien $\varphi(t)$ und $\psi(t)$ zweimal stetig differenzierbare Funktionen, derart daß

$$L_A[\varphi] = 0, \qquad U_A[\varphi] = 0, \tag{4.6}$$

$$\varphi(0) < 0 \quad \text{für} \quad \beta > 0, \qquad \dot{\varphi}(0) < 0 \quad \text{für} \quad \alpha > 0, \tag{4.7}$$

$$L_A[\psi] = 0, \qquad V_A[\psi] = 0,$$

$$\psi(1) < 0 \quad \text{für} \quad \delta > 0, \qquad -\dot{\psi}(1) < 0 \quad \text{für} \quad \gamma > 0,$$

Wie wir unter 4.2 bewiesen haben, kann $\varphi(t)$ wegen (4.6) in einem Punkte t_0 mit $0 \leqq t_0 < 1$ kein relatives Minimum annehmen, für das $\varphi(t_0) < 0$ ist. Mit (4.7) folgt daraus aber

$$\varphi(t) < 0 \begin{cases} \text{für } 0 \leqq t \leqq 1, \text{ falls } \beta > 0, \\ \text{für } 0 < t \leqq 1, \text{ falls } \beta = 0, \end{cases} \quad \dot{\varphi}(t) < 0 \begin{cases} \text{für } 0 \leqq t \leqq 1, \text{ falls } \alpha > 0, \\ \text{für } 0 < t \leqq 1, \text{ falls } \alpha = 0. \end{cases} \tag{4.8}$$

Ebenso sieht man ein:

$$\psi(t) < 0 \begin{cases} \text{für } 0 \leqq t \leqq 1, \text{ falls } \delta > 0, \\ \text{für } 0 \leqq t < 1, \text{ falls } \delta = 0, \end{cases} \quad \dot{\psi}(t) > 0 \begin{cases} \text{für } 0 \leqq t \leqq 1, \text{ falls } \gamma > 0, \\ \text{für } 0 \leqq t < 1, \text{ falls } \gamma = 0. \end{cases} \tag{4.9}$$

Daraus ergibt sich zunächst

$$W(t) = \dot{\varphi}(t)\,\psi(t) - \varphi(t)\,\dot{\psi}(t) > 0 \qquad (0 \leqq t \leqq 1). \tag{4.10}$$

Die Funktion

$$G(t,s) = \frac{1}{p_2(s)}\,\frac{1}{W(s)} \begin{cases} \varphi(t)\,\psi(s) & \text{für} \quad 0 \leqq t \leqq s \leqq 1 \\ \varphi(s)\,\psi(t) & \text{für} \quad 0 \leqq s \leqq t \leqq 1 \end{cases}$$

ist dann die zum Operator A gehörige Greensche Funktion. A^{-1} existiert also und hat die Form

$$A^{-1} v = \int_0^1 G(t, s)\, v(s)\, ds. \tag{4.11}$$

Aus (4.8) bis (4.11) folgt, daß jede Funktion $u = A^{-1} v$ die Eigenschaften (4.5) hat, wenn $v(t) \geqq 0,\ \not\equiv 0$ $(0 \leqq t \leqq 1)$ ist. Das bedeutet aber: A^{-1} besitzt die Eigenschaft (a″) und A daher die Eigenschaft (a).

4.5. Da B offensichtlich die Forderung (1.10) erfüllt, kann man mit Satz 1.1 Aussagen über $A - B$ machen, wenn ein $z \in R$ existiert, für das (1.11) und eventuell (1.12) gilt. Im Falle $p_0(t) > 0$ $(0 \leqq t \leqq 1)$ kann man $P_0 = p_0$ setzen und hat $A - B = A$, erhält also nichts Neues. Es interessiert jetzt, ob $A - B$ auch von monotoner Art ist, wenn $p_0(t)$ auch den Wert 0 oder negative Werte annimmt.

Unter der Voraussetzung (4.4) und

$$\alpha + \gamma > 0 \tag{4.12}$$

erhält man neue Aussagen z. B. mit Hilfe der Funktion

$$\left.\begin{aligned} z_1(t) = \frac{1}{\Delta}\{\beta\,\delta\,\dot{Q}(1) + \alpha\,\gamma\,[Q(1)\,P(t) - Q(t)\,P(1)] + \beta\,\gamma\,[Q(1) - Q(t)] + \\ + \alpha\,\delta\,[Q(1)\,\dot{P}(t) - P(1)\,\dot{Q}(t)]\}, \\ \text{wobei} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ P(t) = \int_0^t p(s)\, ds, \qquad Q(t) = \int_0^t p(s)\,\chi(s)\, ds, \\ p(t) = e^{-\int_0^t \frac{p_1(s)}{p_2(s)}\, ds}, \qquad \chi(t) = \int_0^t \varrho(s)\, ds, \\ \Delta = \beta\,\gamma + \alpha\,\gamma\, P(1) + \alpha\,\delta\, p(1) \end{aligned}\right\} \tag{4.13}$$

ist und $\varrho(t)$ eine stetige Funktion mit

$$\varrho(t) \geqq 0, \ \not\equiv 0 \qquad (0 \leqq t \leqq 1) \tag{4.14}$$

bedeutet.

Man berechnet

$$(A - B)\, z_1(t) = p_2(t)\, p(t)\, \varrho(t) + p_0(t)\, z_1(t), \quad U[z_1] = 0 \quad \text{und} \quad V[z_1] = 0. \tag{4.15}$$

Sei nun zunächst $\varrho(t) \equiv 1$ $(0 \leqq t \leqq 1)$ und

$$p_0(t) \equiv 0 \qquad (0 \leqq t \leqq 1). \tag{4.16}$$

Dann ergibt eine elementare Betrachtung, daß $z_1 \succ o$ ist im Sinne von (4.5), und es gilt ferner $[(A - B) z_1](t) > o$ $(0 \leqq t \leqq 1)$. Mit Satz 1.1 folgt daher, daß im Falle (4.4), (4.12), (4.16) $A - B$ die Eigenschaft (a) hat. Bedeutet also $\varrho(t)$ nun wieder irgendeine stetige Funktion mit (4.14), so ist die Lösung der Aufgabe

$$-p_2\,\ddot{u} - p_1\,\dot{u} = p_2(t)\, p(t)\, \varrho(t), \qquad U[u] = 0, \qquad V[u] = 0$$

$\succ o$ im Sinne von (4.5). Diese Lösung ist jedoch gerade z_1. Im Hinblick auf (4.15) erfüllt also $z = z_1$ aus (4.13), (4.14) die Bedingungen (1.11) und (1.12),

wenn
$$p_2(t)\,p(t)\,\varrho(t) + p_0(t)\,z_1(t) \geqq 0, \;\not\equiv 0 \qquad (0 \leqq t \leqq 1) \tag{4.17}$$
gilt.

Im Falle
$$p_1(t) \equiv 0 \quad (0 \leqq t \leqq 1), \quad \beta = \delta = 0$$
geht (4.17) z.B. über in
$$p_2(t) + p_0(t)\,t(1-t) \geqq 0, \;\not\equiv 0 \quad (0 \leqq t \leqq 1), \quad \text{falls} \quad \varrho(t) \equiv 1 \quad (0 \leqq t \leqq 1),$$
$$\left(p_2(t) + p_0(t)\,\frac{1}{\pi_2}\right) \sin \pi t \geqq 0, \;\not\equiv 0 \quad (0 \leqq t \leqq 1), \quad \text{falls} \quad \varrho(t) = \sin \pi t.$$

Insgesamt liefert Satz 1.1 das folgende Ergebnis.

Satz 4.1a. *Es gelte* (4.4) *und es gebe eine (zweimal stetig differenzierbare) Funktion* $z(t)$, *welche* $\succ o$ *ist im Sinne von* (4.5) *und die Bedingungen*
$$L[z](t) \geqq 0, \;\not\equiv 0 \quad (0 \leqq t \leqq 1), \qquad U[z] = V[z] = 0 \tag{4.18}$$
erfüllt.

Dann folgt aus
$$L[u](t) \geqq 0 \quad (0 \leqq t \leqq 1), \qquad U[u] = V[u] = 0,$$
daß entweder $u \equiv 0$ $(0 \leqq t \leqq 1)$ *ist oder* u *die Eigenschaften* (4.5) *besitzt.*

Im Falle (4.12) *erfüllt* $z = z_1$ *aus* (4.13), (4.14) *die genannten Voraussetzungen, falls* (4.17) *gilt.*

4.6. Man kann auch Aussagen über Funktionen u machen, welche die Randbedingungen $U[u] = V[u] = 0$ nicht erfüllen. Dabei braucht dann auch (4.4) nicht vorausgesetzt zu werden. Wir definieren jetzt:

$R = C[0, 1]$,

S ist die Menge der Kompositionen $v = \langle \varphi(t), \xi, \eta \rangle$ mit $\varphi \in C[0, 1]$ und reellen Zahlen ξ, η,
$$A\,u = \langle L_A[u], U_A[u], V_A[u] \rangle,$$
$$B\,u = \langle (P_0(t) - p_0(t))\,u, (a - \alpha)\,u(0), (c - \gamma)\,u(1) \rangle.$$

Es bedeutet nun
$$u \succ o \quad \text{für} \quad u \in R\colon \quad u(t) > 0 \qquad (0 \leqq t \leqq 1),$$
$$v \succ o \quad \text{für} \quad v = \langle \varphi(t), \xi, \eta \rangle \in S\colon \quad \varphi(t) > 0 \quad (0 \leqq t \leqq 1), \qquad \xi > 0, \quad \eta > 0.$$

Aus 4.2 folgt, daß A die Eigenschaft (c′) besitzt. Eigenschaft (a) läßt sich dagegen nicht allgemein nachweisen. B erfüllt auch hier (1.10).

Satz 1.2 liefert daher jetzt

Satz 4.1b. *Es gebe eine (zweimal stetig differenzierbare) Funktion* $z(t)$, *derart daß*
$$z(t) > 0 \qquad (0 \leqq t \leqq 1), \tag{4.19}$$
$$L[z](t) > 0 \qquad (0 \leqq t \leqq 1), \tag{4.20}$$
$$U[z] > 0, \qquad V[z] > 0 \tag{4.21}$$

gilt. Dann folgt aus

$$L[u](t) \geqq 0 \quad (0 \leqq t \leqq 1), \qquad U[u] \geqq 0, \quad V[u] \geqq 0,$$

daß

$$u(t) \geqq 0 \qquad (0 \leqq t \leqq 1)$$

ist.

Man kann hier z.B. mit der folgenden Funktion $z = z_2$ arbeiten:

$$z_2(t) = z_1(t) + \frac{\varepsilon_1}{\Delta}\{\gamma(P(1) - P(t)) + \delta \dot{P}(1)\} + \frac{\varepsilon_2}{\Delta}\{\alpha P(t) + \beta\}$$

mit den Bezeichnungen (4.13), (4.14) und Konstanten $\varepsilon_1, \varepsilon_2 > 0$. Man hat

$$\begin{gathered} L[z_2](t) = p_2(t)\, p(t)\, \varrho(t) + p_0(t)\, z_2(t) \\ U[z_2] = \varepsilon_1, \qquad V[z_2] = \varepsilon_2. \end{gathered} \tag{4.22}$$

Im Falle (4.4) erfüllt $z = z_2$ die Bedingungen (4.19) und (4.21), und über (4.20) gibt (4.22) Auskunft. Aber auch wenn (4.4) nicht gilt, kann z_2 die gewünschten Eigenschaften haben. Schwierigkeiten macht hier vor allem die Forderung (4.19). Bei $\varepsilon_1 = \varepsilon_2 = \varepsilon$ und $\varrho(t) \equiv 1$ sind z.B. folgende Bedingungen hinreichend für (4.19):

$$\begin{aligned} & p_1(t) \equiv 0 \qquad (0 \leqq t \leqq 1), \\ & \beta = \delta = 1, \\ & \alpha, \gamma > -1, \quad \alpha + \gamma + \alpha\gamma > 0. \end{aligned}$$

L. Collatz zeigt in [*4*] und [*6*] im wesentlichen, daß $L[u](t) \geqq 0$, $U[u] \geqq 0$, $V[u] \geqq 0$ die Aussage $u(t) \geqq 0$ gestattet, falls $p_0(t) \geqq 0$, $\alpha \geqq 0$, $\gamma \geqq 0$, aber nicht $p_0(t) \equiv 0$, $\alpha = 0$ und $\gamma = 0$ ist. Die Beweismethode entspricht der des Abschnittes 4.2. Für selbstadjungierte Aufgaben beweisen einerseits Aronzajn und Smith [*1*] und andererseits R. Bellman [*2*] solche Ergebnisse, wobei $p_0(t)$ auch negative Werte annehmen kann. Aronszajn und Smith arbeiten mit der Methode der reproduzierenden Kerne, Bellman verwendet Mittel der Variationsrechnung. — Auch H. Gloihstehn kann in seiner Arbeit [*8*] negative $p_0(t)$ zulassen. In seinen Ergebnissen kommt eine $z(t)$ entsprechende Funktion vor. Die Resultate in [*8*] werden mit einer anderen, aber ebenfalls elementaren Methode bewiesen.

5. Elliptische und parabolische Differentialgleichungen

5.1. Es sei G ein offenes, beschränktes (nicht notwendig einfach zusammenhängendes) Gebiet des n-dimensionalen reellen Euklidischen Raumes E^n. Die Elemente von E^n werden mit $x = (x^1, x^2, \ldots, x^n)$, $y, \ldots$ bezeichnet. Γ bedeute den Rand von G und $\overline{G}$ die abgeschlossene Mannigfaltigkeit $\overline{G} = G \cup \Gamma$. Γ sei die Vereinigung zweier punktfremder Bestandteile Γ_1 und Γ_2. Γ_2 kann leer sein. Andernfalls sei Γ_2 ein offenes Gebiet einer $(n-1)$-dimensionalen Hyperebene $x^n = \text{const}$. Für jedes $x_0 \in \Gamma_2$ sollen dann außerdem die Punkte

$$x = x_0 - t(0, 0, 0, \ldots, 0, 1) \quad \text{mit} \quad 0 < t \leqq \varepsilon(x_0) \tag{5.1}$$

bei genügend kleinem $\varepsilon(x_0) > 0$ in G liegen.

Ableitungen einer auf $\overline{G}$ erklärten Funktion $u(x)$ nach den Koordinaten x^i werden durch ein Komma und nachfolgende Indizes gekennzeichnet, z.B. $\partial^2 u/\partial x^i \partial x^j = u_{,ij}$. Für $x_0 \in \Gamma_2$ ist $u_{,n}(x_0)$ als einseitige Ableitung aufzufassen.

Wir betrachten den Differentialausdruck

$$L[u] = -\sum_{i,j=1}^{\nu(x)} a_{ij}\, u_{,ij} + \sum_{j=1}^{n} b_j(x)\, u_{,j} + c(x)\, u \quad \text{für} \quad x \in G \cup \Gamma_2$$

mit auf $G \cup \Gamma_2$ definierten gleichmäßig beschränkten Koeffizientenfunktionen. $\nu(x)$ bedeute eine (eventuell von x abhängige) natürliche Zahl, derart daß

$$\nu(x) \begin{cases} \leqq n & \text{für } x \in G, \\ \leqq n-1 & \text{für } x \in \Gamma_2. \end{cases}$$

Für jedes feste $x \in G \cup \Gamma_2$ sei die $\nu(x) \times \nu(x)$-Matrix $\mathfrak{A}(x) = \big(a_{ij}(x)\big)$ symmetrisch und positiv semidefinit. Ist Γ_2 nicht leer, so gelte

$$b_n(x) > 0 \qquad (x \in \Gamma_2). \tag{5.2}$$

Ferner benutzen wir Randausdrücke der Form

$$U[u](x) = \alpha(x)\, u(x) - \beta(x) \frac{\partial u}{\partial \sigma}(x) \quad \text{für} \quad x \in \Gamma_1 \tag{5.3}$$

mit auf Γ_1 definierten gleichmäßig beschränkten Funktionen $\alpha(x)$, $\beta(x)$, derart daß

$$\beta(x) \geqq 0 \qquad (x \in \Gamma_1).$$

$\partial u/\partial\sigma(x_0)$ für $x_0 \in \Gamma_1$ bedeute die einseitige Ableitung in Richtung eines Halbstrahles

$$x = x_0 + \sigma y_0 \qquad (0 \leqq \sigma < \infty) \tag{5.4}$$

mit einem Vektor $y_0(x_0)$ der Länge 1. Für $0 \leqq \sigma \leqq \varepsilon(x_0)$ bei genügend kleinem $\varepsilon(x_0) > 0$ sollen die Punkte dieses Strahles ganz in $G = \overline{G} \cup \Gamma$ liegen.

Wir definieren R_0 als die Menge der auf $\overline{G}$ erklärten Funktionen $u(x)$ mit folgenden Eigenschaften:

a) *$u(x)$ ist stetig auf $\overline{G}$.*

b) *Für $x \in G \cup \Gamma_2$ existieren die Ableitungen*

$$u_{,j}(x) \quad (j = 1, 2, \ldots, n) \quad \text{und} \quad u_{,ij}(x) \quad \big(i, j = 1, 2, \ldots, \nu(x)\big) \tag{5.5}$$

und sind stetig. $L[u](x)$ ist gleichmäßig beschränkt auf $G \cup \Gamma_2$.

c) *In jedem Punkte $x \in \Gamma_1$ mit $\beta(x) \neq 0$ existiert die in* (5.3) *vorkommende Ableitung $\partial u/\partial\sigma(x)$. $U[u](x)$ ist gleichmäßig beschränkt auf Γ_1.*

Jede der in diesem Abschnitt 5 vorkommenden Funktionen $u(x)$ soll $\in R_0$ sein.

Wir wählen eine auf $G \cup \Gamma_2$ definierte gleichmäßig beschränkte Funktion $C(x)$ und eine auf Γ_1 definierte gleichmäßig beschränkte Funktion $a(x)$ mit

$$\begin{aligned} &C(x) \geqq 0, \quad C(x) \geqq c(x) \qquad (x \in G \cup \Gamma_2) \\ &a(x) \geqq 0, \quad a(x) \geqq \alpha(x), \quad a(x) + \beta(x) > 0 \qquad (x \in \Gamma_1), \end{aligned} \tag{5.6}$$

und wir definieren dann

$$L_A[u](x) = -\sum_{i,j=1}^{\nu(x)} a_{ij} u_{,ij} + \sum_{j=1}^{n} b_j u_{,j} + C(x) u,$$
$$U_A[u](x) = \alpha(x) u - \beta(x) u.$$

Die hier in 5.1 genannten Voraussetzungen seien im ganzen Abschnitt 5 erfüllt.

5.2. Wir beweisen zunächst:

Satz 5.1a. *Es sei* $u(x) \in R_0$ *eine Funktion, derart daß*

$$L_A[u](x) > 0 \qquad (x \in G \cup \Gamma_2), \tag{5.7}$$
$$U_A[u](x) > 0 \qquad (x \in \Gamma_1). \tag{5.8}$$

Dann gilt:

$$u(x) > 0 \qquad (x \in \overline{G}).$$

Beweis. Die stetige Funktion $u(x)$ nehme in einem Punkte $x_0 \in \overline{G}$ ihr Minimum an. Wir setzen

$$u(x_0) \leqq 0 \tag{5.9}$$

voraus und leiten daraus einen Widerspruch her.

Zunächst sei $x_0 \in G \cup \Gamma_2$ angenommen. Dann ist

$$u_{,j}(x_0) = 0 \begin{cases} \text{für} \quad j = 1, 2, \ldots, n, & \text{falls} \quad x_0 \in G, \\ \text{für} \quad j = 1, 2, \ldots, n-1, & \text{falls} \quad x_0 \in \Gamma_2. \end{cases} \tag{5.10}$$

Ferner ist die Matrix

$$\mathfrak{U}(x_0) = \left(u_{,ij}(x_0)\right)_{i,j=1,2,\ldots,\nu(x_0)}$$

positiv semidefinit. Dies hat

$$\sum_{i,j=1}^{\nu(x_0)} a_{ij}(x_0)\, u_{,ij}(x_0) \geqq 0 \tag{5.11}$$

zur Folge, denn diese Summe ist die Spur der ebenfalls positiv semidefiniten Matrix $\mathfrak{A}(x_0)\,\mathfrak{U}(x_0)$.

Im Falle $x_0 \in G$ widersprechen (5.9), (5.10) und (5.11) unmittelbar der Voraussetzung (5.7). Im Falle $x_0 \in \Gamma_2$ ergeben (5.9), (5.10), (5.11), (5.7) und (5.2) die Aussage

$$[\partial u / \partial(-x^n)](x_0) < 0,$$

im Widerspruch dazu, daß $u(x_0)$ das Minimum von $u(x)$ sein soll.

Es ist aber auch $x_0 \in \Gamma_1$ ausgeschlossen. Für $\beta(x_0) = 0$ ist dies wegen (5.8) unmittelbar klar. Für $\beta(x_0) > 0$ sagt (5.8) aus, daß $\partial u / \partial \sigma(x_0) < 0$ ist, so daß $u(x_0)$ nicht das Minimum sein kann.

5.3. Um von Satz 5.1a ausgehend mit Satz 1.1 weitere Ergebnisse zu erhalten, definieren wir:

$R = R_0$,

S ist die Menge der Kompositionen $v = \langle \varphi, \omega \rangle$ mit auf $G \cup \Gamma_2$ definierter gleichmäßig beschränkter Funktion $\varphi(x)$ und auf Γ_1 definierter gleichmäßig beschränkter Funktion $\omega(x)$.

Bei der üblichen Ordnungsdefinition bedeutet

$$u \succ o \text{ für } u \in R: \quad u(x) > 0 \qquad (x \in \overline{G}), \tag{5.12}$$

$$v \succ o \text{ für } v = (\varphi, \omega) \in S: \quad \inf\{\varphi(x):\ x \in G \cup \Gamma_2\} > 0, \ \inf\{\omega(x):\ x \in \Gamma_1\} > 0.$$

Ferner seien A und B auf R definiert durch

$$A u = \langle L_A[u], U_A[u] \rangle, \qquad B u = \langle (C(x) - c(x))\, u, (a(x) - \alpha(x))\, u \rangle.$$

Beide Operatoren bilden in S ab.

Satz 5.1a impliziert offenbar, daß der Operator A die Eigenschaft (b) hat. Satz 1.1 liefert daher:

Satz 5.1b. *Es gebe eine Funktion* $z(x) \in R_0$ (s. (5.5)), *derart daß*

$$z(x) > 0 \quad (x \in \overline{G}), \quad \inf\{L[z](x):\ x \in G \cup \Gamma_2\} > 0, \ \inf\{U[z](x):\ x \in \Gamma_1\} > 0. \tag{5.13}$$

Dann folgt aus

$$L[u](x) \geqq 0 \quad (x \in G \cup \Gamma_2), \qquad U[u](x) \geqq 0 \quad (x \in \Gamma_1) \tag{5.14}$$

für $u(x) \in R_0$, *daß*

$$u(x) \geqq 0 \qquad (x \in \overline{G}) \tag{5.15}$$

gilt.

Beispiel mit $n = 2$:

$$G = \{(x^1, x^2):\ -1 \leqq x^1 \leqq 1,\ -1 \leqq x^2 \leqq 1\}, \qquad \Gamma = \Gamma_1$$

$$L[u] = -\Delta u - u,$$

$$U[u] = \begin{cases} u_{,2} & \text{für } x^2 = 1,\ 0 < x^1 < 1, \\ -u_{,2} & \text{für } x^2 = -1,\ 0 < x^1 < 1, \\ u & \text{für } 0 \leqq x^2 \leqq 1,\ x^1 = \pm 1. \end{cases}$$

$z(x^1, x^2) = 10 - 9(x^1)^2 + 2(x^2)^2$ erfüllt (5.13).

5.4. In den Abschnitten 5.4 und 5.5 geben wir für allgemeinere Fälle Funktionen z an, welche (5.13) genügen. Dieser Abschnitt 5.4 bezieht sich im wesentlichen auf elliptische Differentialgleichungen. Wir fordern

$$\Gamma_1 = \Gamma \tag{5.16}$$

und

$$\inf\{\alpha(x):\ x \in \Gamma_1\} > 0 \tag{5.17}$$

und setzen

$$v(x) = n \qquad (x \in G). \tag{5.18}$$

$\lambda_1(x)$ bezeichne den kleinsten Eigenwert der $n \times n$-Matrix $(a_{ij}(x))$. Mit (5.17) beschränken wir uns auf Randoperatoren erster und dritter Art.

Es gibt eine abgeschlossene n-dimensionale Hyperkugel K, derart daß $\overline{G} \subset K$. r_0 sei ihr Radius und r der Abstand von ihrem Mittelpunkt $\tilde{x} = (\tilde{x}^i)$. Man hat $x^i = \tilde{x}^i + r\,\psi^i$ mit

$$\sum_{i=1}^{n} (\psi^i)^2 = 1. \tag{5.19}$$

Wir definieren

$$z_1(x) = z_1[r] = b\,r_0 + \int_r^{r_0} s\, e^{-\int_s^{r_0} \varrho(t)\,dt}\, ds \tag{5.20}$$

mit einer Konstanten b und einer auf $0 \leqq r \leqq r_0$ stetigen Funktion $\varrho(r)$, und zwar wählen wir diese Größen derart, daß

$$\varrho(r) \geqq 0 \qquad (0 \leqq r \leqq r_0) \tag{5.21}$$

und

$$\inf\{\alpha(x)\, b - \beta(x):\ x \in \Gamma_1\} > 0 \tag{5.22}$$

Das ist wegen (5.17) immer möglich.

Es gilt dann $z_1 \succ o$ im Sinne von (5.12). Außerdem ergibt sich in $x \in \Gamma_1$ (mit dem zugehörigen r)

$$\begin{aligned} U[z_1](x) &= \alpha(x)\, z_1(x) + \beta(x)\, r \exp\left[-\int_r^{r_0} \varrho(s)\, ds\right] \cos\bigl(r(x), \sigma(x)\bigr) \\ &\geqq \alpha(x)\, b\, r_0 - \beta(x)\, r_0, \end{aligned}$$

woraus mit (5.22)

$$\inf\{U[z](x):\ x \in \Gamma_1\} > 0$$

folgt.

Schließlich berechnet man

$$L[z_1] = \exp\left[-\int_r^{r_0} \varrho(t)\, dt\right] \left[\sum_{i=1}^{n} a_{ii}(x) + r\left(\varrho(r) \sum_{i,j=1}^{n} a_{ij}\, \psi^i \psi^j + \sum_{j=1}^{n} b_j\, \psi^j\right)\right] + c\, z_1.$$

Hierin kann man mit (5.19) abschätzen:

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \psi^i \psi^j \geqq \lambda_1(x), \qquad \sum_{j=1}^{n} b_j(x)\, \psi^j \leqq \sqrt{\sum_{j=1}^{n} b_j^2(x)} \qquad (x \in G),$$

erhält also

$$L[z](x) \geqq \vartheta(x) \qquad (x \in G)$$

mit

$$\vartheta(x) = \exp\left[-\int_r^{r_0} \varrho(t)\, dt\right] \left\{\sum_{i=1}^{n} a_{ii}(x) + r\left(\varrho(r)\, \lambda_1(x) - \sqrt{\sum_{j=1}^{n} b_j^2(x)}\right)\right\} + c\, z_1(x). \tag{5.23}$$

Satz 5.1 b läßt sich daher ergänzen durch

Satz 5.1 c. *Es gelte* (5.16) *und* (5.17). *Ferner gebe es eine Konstante* b *und eine stetige Funktion* $\varrho(r)$ *mit den Eigenschaften* (5.21), (5.22), *derart daß für die durch* (5.23), (5.20) *definierte Funktion* $\vartheta(x)$

$$\inf\{\vartheta(x):\ x \in G\} > 0 \tag{5.24}$$

gilt.

Dann erfüllt $z = z_1$ *aus* (5.20) *die Voraussetzungen des Satzes* 5.1 b.

Unter den weiter einschränkenden Voraussetzungen

$$c(x) \geqq 0 \qquad (x \in G) \tag{5.25}$$

und

$$\inf\{\lambda_1(x):\ x \in G\} > 0 \tag{5.26}$$

kann man bei gegebener Konstanten b mit (5.22) offenbar immer eine Funktion $\varrho(r)$ so wählen, daß (5.21) und (5.24) gelten. Damit folgt

Satz 5.1d. *Es gelte* (5.16), (5.17), (5.25) *und* (5.26), *und es bedeute* $u \in R_0$ *eine Funktion mit der Eigenschaft*

$$L[u](x) \geqq 0 \quad (x \in G), \qquad U[u](x) \geqq 0 \quad (x \in \Gamma_1).$$

Dann gilt

$$u(x) \geqq 0 \qquad (x \in \overline{G}).$$

Die Voraussetzung (5.26) bedeutet, daß $L[u]$ auf G gleichmäßig elliptisch ist. Diese Voraussetzung wird gar nicht einmal immer benötigt, z.B. nicht im Falle

$$\inf\left\{\sum_{i=1}^{n} a_{ii}(x):\ x \in 0\right\} > 0, \qquad b_j(x) \equiv 0 \qquad (x \in G,\ j = 1, 2, \ldots, n).$$

Satz 5.1d entspricht etwa einem Ergebnis, das L. COLLATZ in [4] und [6] mit Hilfe des Randmaximumsatzes beweist. (Siehe auch A. G. MEYER [11].) — Für selbstadjungierte Differentialgleichungen hat man außerdem wieder Sätze von ARONZAJN und SMITH [1] sowie R. BELLMAN [2], die von ähnlicher Art sind und mit entsprechenden Methoden bewiesen wurden, wie die in Abschnitt 4 erwähnten Ergebnisse bei gewöhnlichen Differentialgleichungen. — H. GLOISTEHN [8] erhält ebenfalls Ergebnisse für elliptische Differentialgleichungen, welche den in Abschnitt 4 erwähnten entsprechen.

5.5. Der folgende Satz bezieht sich auf parabolische Differentialgleichungen. Wir schreiben $t = x^n$.

Satz 5.1e. *Es sei*

$$\nu(x) \leqq n - 1 \qquad (x \in G \cup \Gamma_2), \tag{5.27}$$

und es gebe eine Konstante $N > 0$ *mit der Eigenschaft*

$$\inf\{b_n(x)\, N + c(x):\ x \in G \cup \Gamma_2\} > 0, \tag{5.28}$$

$$\inf\{\alpha(x) - \beta(x) \cos(t, \sigma(x))\, N:\ x \in \Gamma_1\} > 0. \tag{5.29}$$

Dann folgt aus

$$L[u](x) \geqq 0 \qquad (x \in G \cup \Gamma_2), \qquad U[u](x) \geqq 0 \qquad (x \in \Gamma_1),$$

daß

$$u(x) \geqq 0 \qquad (x \in \overline{G})$$

gilt.

Die Richtigkeit der Behauptung ergibt sich daraus, daß $z = e^{Nt}$ die Voraussetzungen des Satzes 5.1e erfüllt.

Die Forderung (5.29) enthält die notwendige Bedingung

$$\inf\{\alpha(x):\ x \in \Gamma_1\} > 0. \tag{5.30}$$

Ist diese Bedingung erfüllt und gilt außerdem

$$\inf\{b_n(x):\ x \in G \cup \Gamma_2\} > 0, \tag{5.31}$$

so gibt es in den folgenden zwei Fällen immer eine Konstante N, für welche (5.28) und (5.29) gelten.

1) Im Falle

$$c(x) \geqq 0 \qquad (x \in G \cup \Gamma_2)$$

ist (5.28) für alle $N > 0$ erfüllt und (5.29) für alle genügend kleinen $N > 0$.

2) Gilt

$$\cos\big(t, \sigma(x)\big) \leqq 0 \quad \text{falls} \quad \beta(x) \neq 0 \qquad (x \in \Gamma_1), \tag{5.32}$$

so ist (5.29) für alle $N > 0$ erfüllt und (5.28) für alle genügend großen $N > 0$.

Die Bedingung (5.32) sagt aus, daß der Randoperator U keine Ableitung längs eines Halbstrahles enthält, welcher eine positive Komponente in t-Richtung besitzt. Da der Differentialoperator $L[u]$ nur die erste Ableitung u_t, nicht aber die zweite Ableitung u_{tt} enthält, erscheint diese Forderung natürlich.

Für parabolische Differentialgleichungen bewiesen K. Nagumo [*12*] und H. Westphal [*18*] einen Satz, der sich auf den Fall $n-1=1$ und die erste Randwertaufgabe bezieht. L. Collatz [*5*] leitete ein entsprechendes Ergebnis für beliebiges n und allgemeinere Randbedingungen auf der „Mantelfläche" eines zylindrischen Gebietes her. (Siehe auch K. Nickel [*13*] und L. Collatz [*6*], S. 329f.; in [*6*] findet man weitere Literaturangaben.) Die Beweismethode dieser Sätze ist elementar, benutzt jedoch spezielle Eigenschaften parabolischer Differentialgleichungen. Es wird dabei eine monotone Art bewiesen, welche der Eigenschaft (b) entspricht. In Satz 5.1e haben wir die bequemere Eigenschaft (c′) verwendet, jedoch sagen unsere Ergebnisse aus, daß (c′) und (b) äquivalent sind.

5.6. Wir skizzieren ergänzend noch einige weitere Überlegungen, ohne auf Einzelheiten einzugehen.

5.61. Wir denken uns Satz 5.1b auf L_A und U_A an Stelle von L und U angewendet. Hat man dann ein z mit den gewünschten Eigenschaften gefunden, so weiß man, daß der Operator A die Eigenschaft (c′) besitzt. Da jedoch der Fall

$$U[u](x) = 0 \qquad (x \in \Gamma_1) \tag{5.33}$$

eingeschlossen ist, sagt Satz 5.1b dann auch aus:

Wenn $\tilde{R} = \{u \in R_0\colon\ U[u](x) = 0 \ \ (x \in \Gamma_1)\}$,

$\tilde{S}$ die Menge der auf $G \cup \Gamma_2$ gleichmäßig beschränkten Funktionen,

$\tilde{A}u = L_A[u]$ und $\tilde{B}u = (C - c)\,u$

ist, hat $\tilde{A}$ die Eigenschaft (c′).

Man kann dann Satz 1.1 auf diese Operatoren $\tilde{A}$ und $\tilde{B}$ anwenden, um weitere Ergebnisse zu bekommen. Dabei kann man mit anderen Funktionen z arbeiten, solchen nämlich, welche die Randbedingungen (5.33) erfüllen, also z.B. Eigenfunktionen und dgl.

Natürlich kann man die Randbedingungen auch etwa nur auf einem Teil von Γ_1 fordern.

5.62. Die Ergebnisse der vorangehenden Nummern sind z.B. von Nutzen bei Abschätzungen für Randwertaufgaben der Form

$$L[u](x) = r(x) \quad (x \in G \cup \Gamma_2), \qquad U[u](x) = \gamma(x) \quad (x \in \Gamma_1).$$

Bei manchen Randwertaufgaben wird nicht in allen Punkten $x \in \Gamma_1$ eine Randbedingung gegeben sein, z. B. nicht in gewissen Unstetigkeitspunkten (etwa Ecken) des Randes. Jedoch werden die Randbedingungen in der Umgebung einer solchen Unstetigkeitsstelle x_0 oft eine Randbedingung der gewünschten Form in x_0 selbst implizieren. Besitzt $u(x)$ in x_0 z.B. eine Tangentialebene und sind alle vorkommenden Funktionen in einer Umgebung von x_0 genügend glatt, so werden die Randbedingungen in der Umgebung von x_0 oft eine lineare Beziehung zwischen $u(x_0)$ und der Stellung der Tangentialebene festlegen, und damit auch eine lineare Beziehung zwischen $u(x_0)$ und einer Ableitung $\partial u/\partial\sigma(x_0)$ in einer geeigneten Richtung. Entsprechende Differenzierbarkeitseigenschaften sind dann zusätzlich von den Funktionen $u \in R_0$ zu fordern.

5.63. Wir nehmen

$$C(x) = c(x) \quad (x \in G \cup \Gamma_2), \qquad a(x) = \alpha(x) \quad (x \in \Gamma_1)$$

an. Dann sagt Satz 5.1b aus, daß A die Eigenschaft (c′) hat, wenn man ein geeignetes z finden kann. Um ein solches z zu konstruieren, mußten wir vielfach Randoperatoren zweiter Art ausschließen. Wir zeigen nun, wie man die Eigenschaft (c′) auch ohne die Voraussetzung $a(x) > 0$ unter einigen zusätzlichen Annahmen direkt nachweisen kann. Diese Überlegungen haben mehr theoretisches Interesse, da man bei der Fehlerabschätzung gemäß Satz 1.2 praktisch ohnehin eine Funktion z ermitteln muß.

Es sei jetzt

$$\inf\{C(x)\colon\ x \in G\} > 0. \tag{5.34}$$

Dann läßt sich sehr einfach beweisen, daß $u(x)$ in $G \cup \Gamma_2$ kein negatives Minimum haben kann, wenn

$$L_A[u](x) \geqq 0 \qquad (x \in G \cup \Gamma_2)$$

ist, und daß auch in einem Punkte $x_0 \in \Gamma_1$ kein negatives Minimum vorliegen kann, falls

$$U_A[u](x_0) \geqq 0 \quad \text{und} \quad \alpha(x_0) > 0$$

gilt. Die Beweismethode ist ganz ähnlich wie in Abschnitt 5.2.

Wir zeigen nun unter geeigneten Voraussetzungen: Im Falle

$$L[u](x) \geqq 0 \qquad (x \in G \cup \Gamma_2), \tag{5.35}$$

$$U[u](x_0) \geqq 0 \quad \text{für ein } x_0 \in \Gamma_1, \quad \alpha(x_0) = 0, \tag{5.36}$$

$$u(x_0) < 0 \tag{5.37}$$

kann $u(x_0)$ nicht das Minimum von $u(x)$ sein. Zum Beweis nehmen wir an, $u(x_0)$ sei das Minimum. Dann folgt mit (5.36) sofort

$$u_\sigma(x_0) = 0. \tag{5.38}$$

Um einen Widerspruch herzuleiten, benötigen wir neben (5.34) weitere einschränkende Voraussetzungen. $\xi^1, \xi^2, \ldots, \xi^{n-1}, \sigma$ mögen die Koordinaten eines rechtwinkligen Koordinatensystems mit Ursprung in x_0 bedeuten. Es gebe Konstanten $\varepsilon, \delta > 0$, derart daß die durch

$$\varepsilon^2 \sigma - \delta (\xi)^2 \geqq 0, \qquad 0 < \sigma \leqq \delta \quad \text{mit} \quad (\xi)^2 = (\xi^1)^2 + (\xi^2)^2 + \cdots + (\xi^{n-1})^2$$

beschriebene Punktmenge P ganz in G liegt. Wenn Γ und $u(x)$ (in einer Umgebung von x_0) genügend glatt sind, kann man folgende Schlüsse ziehen.

Wegen (5.34), (5.35) und (5.37) gilt

$$\sum_{i,j=1}^{\nu(x)} a_{ij}(x)\, u_{,ij} - \sum_{j=1}^{n} b_j(x)\, u_{,j} \leqq C(x)\, u \leqq -3N \qquad (x \in G) \tag{5.39}$$

mit einer Konstanten $N > 0$.

Wenn $u(x)$ als Funktion auf Γ_1 in x_0 differenzierbar ist, verschwinden in x_0 alle Ableitungen von u in Richtung von Γ_1. Außerdem gilt aber (5.38), wobei σ ins Innere von G gezählt wird. Verhalten sich die Ableitungen $u_{,j}(x)$ $(j = 1, 2, \ldots, n)$ bei Annäherung $x \to x_0$ stetig, so gilt also bei genügend klein gewähltem δ

$$\left| \sum_{j=1}^{n} b_j(x)\, u_{,j} \right| \leqq N \qquad (x \in P) \tag{5.40}$$

und damit

$$\sum_{i,j=1}^{\nu(x)} a_{ij}(x)\, u_{,ij} \leqq -2N < 0 \qquad (x \in P). \tag{5.41}$$

Wir definieren nun

$$w(x) = u(x) + v(x) - u(x_0),$$

wobei

$$v(x) = \varkappa \big((\xi)^2 \delta - \sigma \varepsilon^2 \big)$$

ist mit einer Konstanten $\varkappa > 0$.

δ sei so klein gewählt, daß (5.40) gilt. Bei hinreichend kleinem $\varkappa$ ist

$$\left| \sum_{i,j=1}^{\nu(x)} a_{ij}\, v_{,ij} \right| \leqq N \qquad (x \in P),$$

also wegen (5.41)

$$\sum_{i,j=1}^{\nu(x)} a_{ij}\, w_{,ij} \leqq -N < 0 \qquad (x \in P). \tag{5.42}$$

Ferner gilt $u(x) > u(x_0)$ für $x \in G$, da $u(x)$ für $x \in G$ kein negatives Minimum annehmen kann. Bei genügend kleinem $\varkappa$ hat man daher $w(x) \geqq 0$ auf dem ganzen Rande von P. Außerdem berechnete man mit (5.38)

$$\frac{\partial w}{\partial \sigma}(x_0) = \frac{\partial v}{\partial \sigma}(x_0) = -\varkappa\, \varepsilon^2 < 0.$$

w nimmt im Innern von P also negative Werte an und muß daher dort ein Minimum besitzen. Dies widerspricht jedoch (5.42).

5.64. Zum Schluß des Abschnittes 5 sei noch eine gewisse Ergänzung des Randmaximumsatzes (bzw. Randminimumsatzes) bewiesen. Sie bezieht sich auf Differentialgleichungen der allgemeinen Form in Abschnitt 5.1, also sowohl auf elliptische als auch auf parabolische Differentialgleichungen.

Satz 5.2. *Es existiere eine Funktion* $z(x) \in R_0$, *welche den Beziehungen*

$$\inf\{z(x):\ x \in \Gamma_1\} > 0, \qquad \inf\{L[z](x):\ x \in G \cup \Gamma_2\} > 0$$

genügt. u bedeute eine Funktion $\in R_0$, *derart daß*

$$L[u](x) \geqq 0 \qquad (x \in G \cup \Gamma_2)$$

und

$$\mu \cdot c(x) \leqq 0 \qquad (x \in G \cup \Gamma_2)$$

mit

$$\mu = \inf\{u(y):\ y \in \Gamma_1\}.$$

Dann ist

$$u(x) \geqq \mu \qquad (x \in \bar{G}). \tag{5.43}$$

Beweis. Für die Funktion $v(x) = u(x) - \mu$ hat man

$$L[v](x) = L[u](x) - c\mu \geqq L[u](x) \geqq 0 \qquad (x \in G \cup \Gamma_2)$$

und

$$v(x) \geqq 0 \qquad (x \in \Gamma_1).$$

Satz 5.1 b ergibt daher die Aussage $v(x) \geqq 0$ $(x \in \bar{G})$, welche (5.43) äquivalent ist.

6. Ergänzungen und Verallgemeinerungen

6.1. Mit der Verallgemeinerung einer Methode, welche L. COLLATZ in [*4*] bei Differentialgleichungen anwendete, lassen sich aus den Ergebnissen für lineare Probleme Resultate für nichtlineare Aufgaben herleiten.

Es sei T ein nichtlinearer Operator, der aus einem linearen Raum $\hat{R}$ in einen linearen Raum S abbildet. Für bestimmte u, v lasse sich die Differenz $Tu - Tv$ in der Form

$$Tu - Tv = M(u, v)\,(u - v) \tag{6.1}$$

darstellen. Dabei bedeute $M(u, v)$ einen (von u und v abhängigen) linearen Operator, welcher eine lineare Teilmenge $R \subset \hat{R}$ in S abbildet. R und S mögen die in Abschnitt 1 geforderten Eigenschaften besitzen. Ist $M(u, v)$ für bestimmte u, v von monotoner Art im Sinne einer der Eigenschaften in (1.4) und (1.5), so folgt aus (6.1), daß T im Bereich dieser Elemente in entsprechendem Sinne von monotoner Art ist.

Im einfachsten Falle ist M von u und v unabhängig. Wir nennen T dann affin. Dieser Fall liegt z.B. bei „linearen" inhomogenen Randwertaufgaben vor. Für einen affinen Operator T läßt sich mit Satz 1.1 leicht ein Satz 1.2 entsprechendes Ergebnis herleiten, das die Fehlerabschätzung für eine Gleichung $Tu = r$ direkt mit dem Beweis der monotonen Art von T verknüpft.

6.2. In Abschnitt 1 haben wir mit den Ordnungsbeziehungen $\geqq$, $>$, $\succ$ und $\succeq$ gearbeitet. Sind die Elemente $v \in S$ (oder auch die $u \in R$) z.B. Kompositionen $v = (v^1, \ldots, v^p)$ mit Elementen v^i aus archimedischen halbgeordneten linearen Räumen S_i, so kann man Ordnungsrelationen betrachten, welche für die einzelnen Elemente v^i verschiedene Beziehungen benutzen. Zum Beispiel wird im Falle $p = 2$ durch

$$v^1 \geqq o, \qquad v^2 \succ o$$

eine bestimmte Art von „Positivität" des Elementes $v=(v^1, v^2)$ beschrieben. Es lassen sich dann Satz 1.1 und Satz 1.2 entsprechende Ergebnisse gewinnen. So kann man vorgehen, um einige von H. GLOISTEHN [*8*] für Differentialgleichungen bewiesene Resultate auf abstraktem Wege herzuleiten. Darauf sei hier jedoch nicht näher eingegangen.

6.3. Für hyperbolische Differentialgleichungen geben wir hier nur ein spezielles Beispiel. Und zwar betrachten wir den Differentialausdruck

$$L[u]=u_{xy}-a(x,y)\,u_x-b(x,y)\,u_y-c(x,y)\,u$$

im Gebiet

$$G=\{(x,y):\ 0\leqq x\leqq 1,\ 0\leqq y\leqq 1\}.$$

a, b und c seien auf G stetige Funktionen, derart daß

$$a(x,y)\geqq 0,\quad b(x,y)\geqq 0,\quad c(x,y)\geqq 0\quad\text{für}\quad (x,y)\in G. \tag{6.2}$$

Alle im folgenden vorkommenden Funktionen $u(x,y)$ seien mitsamt ihren Ableitungen u_{xy}, x_x und u_y als stetig auf G vorausgesetzt.

Satz 6.1. *Aus*

$$L[u](x,y)\geqq 0 \qquad (0\leqq x,\ y\leqq 1)$$

$$u(x,0)=0\quad(0\leqq x\leqq 1)\quad\textit{und}\quad u(0,y)=0\quad(0\leqq y\leqq 1) \tag{6.3}$$

folgt

$$u(x,y)\geqq 0,\quad u_x(x,y)\geqq 0\quad\textit{und}\quad u_y(x,y)\geqq 0\qquad(0\leqq x,\ y\leqq 1). \tag{6.4}$$

Beweis. Es sei R die Menge der auf G definierten Funktionen $u(x,y)$, welche die erwähnten Differenzierbarkeitseigenschaften besitzen und den (charakteristischen) Anfangsbedingungen (6.3) genügen. S bedeute den Raum der auf G stetigen Funktionen. In S verwenden wir die übliche Ordnungsdefinition, in R dagegen erklären wir $u\geqq o$ durch (6.4).

Der durch $Au=u_{xy}$ auf R definierte Operator hat dann die Eigenschaft (c'), und für den durch $Bu=a(x,y)\,u_x+b(x,y)\,u_y+c(x,y)\,u$ gegebenen Operator B gilt (1.10) wegen (6.2). Die Funktion $z(x,y)=x\,y\,e^{N(x+y)}$ is $\succ o$ als Element $\in R$ und erfüllt (1.13) für eine genügend große Konstante N. Daher folgt Satz 6.1 mit Satz 1.1.

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A Mixed Boundary-Value Problem for Hyperbolic First-Order Systems with Derivatives in the Boundary Conditions

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This paper* is concerned with a modification of the customary boundary-value problem of mixed type for first-order hyperbolic systems of partial differential equations in two independent variables. The modification consists in that first-order derivatives, and not only the unknown functions themselves, appear in the boundary conditions. Boundary-value problems of this type occur in gas dynamics, as was pointed out to the author by GLENN LEWIS.

This paper is to be considered as an immediate continuation of the previous paper [*1*]. The notation is not re-explained here, and arguments coinciding with arguments in [*1*] are not repeated in detail.

1. The boundary-value problem

We consider the linear hyperbolic operator

$$\mathscr{L} u = \frac{\partial u}{\partial t} - \mathscr{D}\frac{\partial u}{\partial x} + \mathscr{G} u, \qquad \mathscr{L} \in \mathfrak{L}\left(\mathscr{C}^1_m(\mathscr{V}), \mathscr{C}^0_m(\mathscr{V})\right) \tag{1.1}$$

in the domain $\mathscr{V} = \{(x, t) \mid x \geqq 0,\ t \geqq 0,\ x + t/\lambda \leqq 1\}$ in Euclidean 2-space. Here $\mathscr{D}$ is a real diagonal matrix $\in \mathscr{C}^0_{m\times m}(\mathscr{V})$ of the form

$$\mathscr{D} = \begin{pmatrix} \mathscr{D}^+ & 0 \\ 0 & \mathscr{D}^- \end{pmatrix} \tag{1.2}$$

where $\mathscr{D}^+ \in \mathscr{C}^0_{m^+\times m^+}(\mathscr{V})$ and $\mathscr{D}^- \in \mathscr{C}^0_{m^-\times m^-}(\mathscr{V})$ are positive and negative in $\mathscr{V}$, respectively, and $\mathscr{G} \in \mathscr{C}^0_{m\times m}(\mathscr{V})$. We assume

$$0 < \lambda \leqq |\mathscr{D}|^{-1}_{\mathscr{V}}.$$

Corresponding to the partitioning (1.2) of $\mathscr{D}$, we can write (1.1) as

$$\mathscr{L}^+ u = \frac{\partial u^+}{\partial t} - \mathscr{D}^+ \frac{\partial u^+}{\partial x} + \mathscr{G}^+ u,$$

$$\mathscr{L}^- u = \frac{\partial u^-}{\partial t} - \mathscr{D}^- \frac{\partial u^-}{\partial x} + \mathscr{G}^- u$$

$\left(u = (u^+, u^-),\ u^+ \in \mathscr{C}^0_{m^+}(\mathscr{V}),\ u^- \in \mathscr{C}^0_{m^-}(\mathscr{V}),\ \mathscr{G}^+ \in \mathscr{C}^0_{m\times m^+}(\mathscr{V}),\ \mathscr{G}^- \in \mathscr{C}^0_{m\times m^-}(\mathscr{V})\right)$.

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With ϱ^0 and ϱ^- being the restriction operators to $\mathscr{S}^0 = \mathscr{V} \cap \{t=0\}$ and $\mathscr{S}^- = \mathscr{V} \cap \{x=0\}$, we have the initial-value operator

$$\ell^0 u = \varrho^0 u\,, \qquad \ell^0 \in \mathfrak{L}\left(\mathscr{C}_m^0(\mathscr{V}),\, \mathscr{C}_m^0(\mathscr{S}^0)\right).$$

We partition u^- into $(u^{\overset{1}{-}}, u^{\overset{2}{-}})$, $u^{\overset{i}{-}} \in \mathscr{C}^0_{m^{\overset{i}{-}}}(\mathscr{V})$, $i=1, 2$. On $\mathscr{S}^-$ we have the boundary operator $\ell^- u = (\ell^{\overset{1}{-}} u, \ell^{\overset{2}{-}} u) \in \mathfrak{L}\left(\mathscr{C}_m^1(\mathscr{V}),\, \mathscr{C}_m^0(\mathscr{S}^-)\right)$, where

$$\begin{aligned} \ell^{\overset{1}{-}} u &= \varrho^- \frac{\partial u^{\overset{1}{-}}}{\partial t} + \mathscr{B}^1 \varrho^- u\,, && \ell^{\overset{1}{-}} \in \mathfrak{L}\left(\mathscr{C}_m^1(\mathscr{V}),\, \mathscr{C}^0_{m^{\overset{1}{-}}}(\mathscr{S}^-)\right), \\ \ell^{\overset{2}{-}} u &= \mathscr{B}^2 \varrho\ \ u\,, && \ell^{\overset{2}{-}} \in \mathfrak{L}\left(\mathscr{C}_m^1(\mathscr{V}),\, \mathscr{C}^0_{m^{\overset{2}{-}}}(\mathscr{S}^-)\right). \end{aligned}$$

Here

$$\mathscr{B}^1 \in \mathscr{C}^0_{m \times m^{\overset{1}{-}}}(\mathscr{S}^-)\,, \qquad \mathscr{B}^2 = (\mathscr{B}^{2+}, \mathscr{B}^{2\overset{1}{-}}, I^{\overset{2}{-}}) \in \mathscr{C}^0_{m \times m^{\overset{2}{-}}}(\mathscr{S}^-)\,,$$

where $I^{\overset{2}{-}}$ is the identity matrix of type $m^{\overset{2}{-}} \times m^{\overset{2}{-}}$.

The boundary-value problem we consider is the following: Given $\mathscr{F}, f^0, f$, find u such that

$$\begin{aligned} \mathscr{L} u &= \mathscr{F}\,, \\ \ell^0 u &= f^0, \qquad\qquad (1.3) \\ \ell^- u &= f^-. \end{aligned}$$

2. The discrete boundary-value problem

In the same way as in [*1*], we approximate $\mathscr{L}$ by means of the difference operators

$$\begin{aligned} L^+ u &= \varDelta_\tau u^+ - D^+ \varDelta_\xi^+ u^+ + G^+ u\,, && L^+ \in \mathfrak{L}\left(\mathscr{C}_m(V),\, \mathscr{C}_{m^+}(V_1)\right), \\ L^- u &= \varDelta_\tau u^- - D^- \varDelta_\xi^- u^- + G^- u\,, && L\ \ \in \mathfrak{L}\left(\mathscr{C}_m(V),\, \mathscr{C}_{m^-}(V_1^-)\right) \end{aligned}$$

(for notation, *cf.* [*1*]).

The discrete initial-value operator is

$$l^0 u = \sigma^0 u\,, \qquad l^0 \in \mathfrak{L}\left(\mathscr{C}_m(V),\, \mathscr{C}_m(S^0)\right).$$

Corresponding to $\mathscr{S}^-$ we introduce the notation

$$\begin{aligned} S^{\overset{1}{-}} &= \{(0, \tau k) \mid \tau = 0, \ldots, n\}, \\ S^{\overset{2}{-}} &= \{(0, \tau k) \mid \tau = 1, \ldots, n\}, \\ S_p^{\overset{i}{-}} &= V_p \cap S^{\overset{i}{-}}, \qquad i = 1, 2 \end{aligned}$$

and the discrete boundary operators

$$\begin{aligned} l^{\overset{1}{-}} u &= \sigma^{\overset{1}{-}} \varDelta_\tau u^{\overset{1}{-}} + B^1 \sigma^{\overset{1}{-}} u\,, && l^{\overset{1}{-}} \in \mathfrak{L}\left(\mathscr{C}_m(V),\, \mathscr{C}_{m^{\overset{1}{-}}}(S^{\overset{1}{-}})\right), \\ l^{\overset{2}{-}} u &= B^2 \sigma^{\overset{2}{-}} u\,, && l^{\overset{2}{-}} \in \mathfrak{L}\left(\mathscr{C}_m(V),\, \mathscr{C}_{m^{\overset{2}{-}}}(S^{\overset{2}{-}})\right), \end{aligned}$$

where $B^i = r^{\overset{i}{-}} \mathscr{B}^i$, $r^{\overset{i}{-}}$ and $\sigma^{\overset{i}{-}}$ being the restriction operators from $\mathscr{S}^-$ and V to $S^{\overset{i}{-}}$. In addition to the boundary-value problem (1.3), we shall consider the following discrete boundary-value problem: Given $\mathscr{F} = (\mathscr{F}^+, \mathscr{F}^-) \in \mathscr{C}_m^0(\mathscr{V})$, $f^0 \in \mathscr{C}_m^0(\mathscr{S}^0)$,

$\mathscr{f}^- = (\mathscr{f}^{\overset{1}{-}}, \mathscr{f}^{\overset{2}{-}}) \in \mathscr{C}^0_{m^-}(\mathscr{S}^-)$, find $u \in \mathscr{C}_m(V)$ such that

$$
\begin{aligned}
&L^+ u = F^+, &&\text{where} \quad F^+ = R_1 \mathscr{F}^+,\\
&L^- u = F^-, &&\text{where} \quad F^- = R_1^- \mathscr{F}^-,\\
(2.1) \qquad &l^0 u = f^0, &&\text{where} \quad f^0 = r^0 \mathscr{f}^0,\\
&l^{\overset{1}{-}} u = f^{\overset{1}{-}}, &&\text{where} \quad f^{\overset{1}{-}} = r^{\overset{1}{-}} \mathscr{f}^{\overset{1}{-}},\\
&l^{\overset{2}{-}} u = f^{\overset{2}{-}}, &&\text{where} \quad f^{\overset{2}{-}} = r^{\overset{2}{-}} \mathscr{f}^{\overset{2}{-}}.
\end{aligned}
$$

In the same way as in [*1*] we have the following theorem:

Theorem 2.1. *There is one and only one solution $u \in \mathscr{C}_m(V)$ of the discrete boundary-value problem* (2.1).

The solution of (2.1) can be determined recursively from the following explicit scheme, which is obtained directly from the definitions

$$
\begin{aligned}
& u_{\xi,0} = l^0 u_\xi, && \xi = 0, 1, \dots, n;\\
& u^+_{\xi,\tau+1} = \lambda D^+_{\xi,\tau} u^+_{\xi+1,\tau} + (I^+ - \lambda D^+_{\xi,\tau})\, u^+_{\xi,\tau} - \\
& \qquad - k G^+_{\xi,\tau} u_{\xi,\tau} + k L^+ u_{\xi,\tau}, && \xi = 0, 1, \dots, n-\tau-1;\\
(2.2) \qquad & u^-_{\xi,\tau+1} = (I^- + \lambda D^-_{\xi,\tau})\, u^-_{\xi,\tau} - \lambda D^-_{\xi,\tau} u^-_{\xi-1,\tau} - \\
& \qquad - k G^-_{\xi,\tau} u_{\xi,\tau} + k L^- u_{\xi,\tau}, && \xi = 1, \dots, n-\tau-1;\\
& u^{\overset{1}{-}}_{0,\tau+1} = u^{\overset{1}{-}}_{0,\tau} - k B^{\overset{1}{-}}_\tau u_{0,\tau} + k l^{\overset{1}{-}} u_\tau;\\
& u^{\overset{2}{-}}_{0,\tau+1} = l^{\overset{2}{-}} u_{\tau+1} - B^{2+}_{\tau+1} u^+_{0,\tau+1} - B^{2\overset{1}{-}}_{\tau+1} u^{\overset{1}{-}}_{0,\tau+1}.
\end{aligned}
$$

3. Estimates

Theorem 3.1. *There is a constant C, independent of n and of $u \in \mathscr{C}_m(V)$, such that*

$$|u|_V \leqq C\{|L^+ u|_{V_1} + |L^- u|_{V_1^-} + |l^0 u|_{S^0} + |l^{\overset{1}{-}} u|_{S_1^{\overset{1}{-}}} + |l^{\overset{2}{-}} u|_{S^{\overset{2}{-}}}\}.$$

Proof. Put

$$
\begin{aligned}
a^+_\tau &= \max\left(|u^+|_{S^\tau}, |u^{\overset{1}{-}}|_{S^\tau}\right),\\
a^-_\tau &= |u^{\overset{2}{-}}|_{S^\tau},\\
a_\tau &= \max(a^+_\tau, a^-_\tau),\\
K &= |L^+ u|_{V_1} + |L^- u|_{V_1^-} + |l^{\overset{1}{-}} u|_{S_1^{\overset{1}{-}}} + |l^{\overset{2}{-}} u|_{S^{\overset{2}{-}}}.
\end{aligned}
$$

Then, because of the boundedness of the coefficients in the scheme (2.2) and the condition $0 < \lambda \leqq |\mathscr{D}|^{-1}_{\mathscr{V}}$, we obtain

$$
\begin{aligned}
& a^+_{\tau+1} \leqq a^+_\tau + n^{-1} C_1 (a_\tau + K), && \tau = 0, 1, \dots, n-1,\\
& a^-_{\tau+1} \leqq \max\{a^-_\tau + n^{-1} C_2 (a_\tau + K);\ C_3 (a^+_{\tau+1} + K)\}, && \tau = 0, 1, \dots, n-1.
\end{aligned}
$$

The theorem therefore follows from Lemma 4.2 in [*1*].

In this way we can also estimate difference quotients and obtain the following theorem.

Theorem 3.2. *Assume that* $\mathscr{D}, \mathscr{G}, \mathscr{B}^1, \mathscr{B}^2 \in \mathscr{C}^q$. *Then there is a constant* C, *independent of* n *and of* $u \in \mathscr{C}_m(V)$, *such that*

$$|u|_{V,q} \leqq C\{|L^+ u|_{V_1,q} + |L^- u|_{V_1^-,q} + |l^0 u|_{S^0,q} + |l^{\overset{1}{-}} u|_{S^{\overset{1}{-}}_1,q} + |l^{\overset{2}{-}} u|_{S^{\overset{2}{-}},q} + |u^-|_{0,q}\}.$$

Just as in [*1*] we can obtain the following theorem in the limit.

Theorem 3.3. *Assume that* $\mathscr{D}, \mathscr{G}, \mathscr{B}^1, \mathscr{B}^2 \in \mathscr{C}^q$; *then there is a constant* C, *independent of* $u \in \mathscr{C}_m^{q+1}(\mathscr{V})$, *such that*

$$|u|_{\mathscr{V},q} \leqq C\{|\mathscr{L} u|_{\mathscr{V},q} + |\ell^0 u|_{\mathscr{S}^0,q} + |\ell^- u|_{\mathscr{S}^-,q}\}.$$

4. Compatibility conditions

We say that $\mathscr{F} \in \mathscr{C}_m^{q-1}(\mathscr{V})$, $f^0 \in \mathscr{C}_m^q(\mathscr{S}^0)$, $f^{\overset{1}{-}} \in \mathscr{C}^{q-1}_{m^{\overset{1}{-}}}(\mathscr{S}^-)$, $f^{\overset{2}{-}} \in \mathscr{C}^{q}_{m^{\overset{2}{-}}}(\mathscr{S}^-)$ satisfy compatibility conditions of order q at the origin (for the boundary-value problem (1.3)) if there is a $\bar{u} \in \mathscr{C}_m^q(\mathscr{V})$ such that

$$\begin{aligned}
&\partial^\alpha(\mathscr{L}\bar{u})(0,0) = \partial^\alpha \mathscr{F}(0,0), && |\alpha| \leqq q-1, && \\
&d^\beta(\ell^0 \bar{u})(0) \quad = d^\beta f^0(0), && \beta \leqq q && \left(d = \frac{d}{dx}\right), \\
&d^\beta(\ell^{\overset{1}{-}} \bar{u})(0) \quad = d^\beta f^{\overset{1}{-}}(0), && \beta \leqq q-1 && \left(d = \frac{d}{dt}\right), \\
&d^\beta(\ell^{\overset{2}{-}} \bar{u})(0) \quad = d^\beta f^{\overset{2}{-}}(0), && \beta \leqq q && \left(d = \frac{d}{dt}\right).
\end{aligned} \tag{4.1}$$

It is of course a necessary condition for the existence of a solution in $\mathscr{C}_m^q(\mathscr{V})$ of the boundary-value problem (1.3) that $\mathscr{F}, f^0, f^{\overset{1}{-}}, f^{\overset{2}{-}}$ satisfy compatibility conditions of order q at the origin. We have the following theorem:

Theorem 4.1. *There are matrices* $M^i_{\gamma\beta}$ *and* $N^i_{\gamma\alpha}$, *which depend only on* $\mathscr{D}, \mathscr{G}$, $\mathscr{B}^1$, *and* $\mathscr{B}^2$, *such that if* $\mathscr{F}, f^0, f^{\overset{1}{-}}$, *and* $f^{\overset{2}{-}}$ *satisfy compatibility conditions of order* q *at the origin, then for* $\gamma \leqq q-1$ *and* $\gamma \leqq q$ *in the cases* $i = 1$ *and* 2, *respectively,*

$$\begin{aligned}
d^\gamma f^{\overset{1}{-}}(0) &= \sum_{\beta=0}^{\gamma+1} M^1_{\gamma\beta}\, d^\beta f^0(0) + \sum_{|\alpha| \leqq \gamma} N^1_{\gamma\alpha}\, \partial^\alpha \mathscr{F}(0,0), \\
d^\gamma f^{\overset{2}{-}}(0) &= \sum_{\beta=0}^{\gamma} M^2_{\gamma\beta}\, d^\beta f^0(0) + \sum_{|\alpha| \leqq \gamma-1} N^2_{\gamma\alpha}\, \partial^\alpha \mathscr{F}(0,0),
\end{aligned} \tag{4.2}$$

and such that if on the other hand (4.2) *is satisfied for* $\mathscr{F}, f^0, f^{\overset{1}{-}}$, *and* $f^{\overset{2}{-}}$, *then we can find a polynomial* $\bar{u} = \bar{u}(x,t)$ *of degree* q *such that the relations* (4.1) *are satisfied.*

In particular, for $q = 1$ the matrices M and N are given in the following theorem:

Theorem 4.2. *Assume that* $\mathscr{B}^2 \in \mathscr{C}^1$. *Necessary conditions that the boundary-value problem* (1.3) *have a solution in* $\mathscr{C}_m^1(\mathscr{V})$ *are the relations*

$$f^{\overset{1}{-}}(0) = \mathscr{D}^{\overset{1}{-}}(0,0)\, d f^{0\overset{1}{-}}(0) - \mathscr{G}^{\overset{1}{-}}(0,0)\, f^0(0) + \mathscr{F}^{\overset{1}{-}}(0,0) + \mathscr{B}^1(0)\, f^0(0), \tag{4.3}$$

$$f^{\overset{2}{-}}(0) = \mathscr{B}^2(0)\, f^0(0), \tag{4.4}$$

$$d f^{\overset{2}{-}}(0) = \mathscr{B}^2(0)\,[\mathscr{D}(0,0)\, d f^0(0) - \mathscr{G}(0,0)\, f^0(0) + \mathscr{F}(0,0)] + d\mathscr{B}^2(0)\, f^0(0). \tag{4.5}$$

In order to see that conditions (4.3), (4.4), and (4.5) are also essentially sufficient compatibility conditions for the boundary-value problem (1.3) we prove the following lemma:

Lemma 4.1. *Assume that* $\mathscr{D}, \mathscr{G}, \mathscr{B}^1, \mathscr{F}, \mathscr{f}^{\overset{1}{-}} \in \mathscr{C}^1$; $\mathscr{B}^2, \mathscr{f}^0, \mathscr{f}^{\overset{2}{-}} \in \mathscr{C}^2$, *and that the conditions* (4.3), (4.4), *and* (4.5) *are satisfied. Then there is a constant* C *such that if* u_n *is the solution of the discrete boundary-value problem* (2.1), *then, for all* n, $|u_n^-|_{0,2} \leqq C$.

Proof. We first remark that $\Delta_\xi^k u_{0,0} = \Delta_\xi^k f_0^0$, $k=0, 1, 2$, are bounded since $\mathscr{f}^0 \in \mathscr{C}^2$. We have

$$\Delta_\tau u_{0,0}^+ = D_{0,0}^+ \Delta_\xi f_0^{0+} - G_{0,0}^+ f_0^0 + F_{0,0}^+,$$
$$\Delta_\tau u_{0,0}^{\overset{1}{-}} = f_0^{\overset{1}{-}} - B_0^1 f_0^0,$$

and thus $\Delta_\tau u_{0,0}^+$ and $\Delta_\tau u_{0,0}^{\overset{1}{-}}$ are bounded. We have

$$u_{0,\tau}^{\overset{2}{-}} = f_\tau^{\overset{2}{-}} - B_\tau^{2+} u_{0,\tau}^+ - B_\tau^{2\overset{1}{-}} u_{0,\tau}^{\overset{1}{-}}, \qquad \tau = 0, 1, 2.$$

In case $\tau=0$, this follows from (4.4); in case $\tau=1, 2$, from (2.2). Therefore

$$\Delta_\tau u_{0,0}^{\overset{2}{-}} = \Delta_\tau (f_\tau^{\overset{2}{-}} - B_\tau^{2+} u_{0,\tau}^+ - B_\tau^{2\overset{1}{-}} u_{0,\tau}^{\overset{1}{-}})_0$$

is bounded because of the regularity assumptions and the boundedness of $\Delta_\tau u_{0,0}^+$ and $\Delta_\tau u_{0,0}^{\overset{1}{-}}$, already proved. We have

$$\begin{aligned}
\Delta_\xi \Delta_\tau u_{0,0}^+ &= \Delta_\xi (D_{\xi,0}^+ \Delta_\xi u_{\xi,0}^+ - G_{\xi,0}^+ u_{\xi,0} + F_{\xi,0}^+)_0, \\
\Delta_\tau^2 u_{0,0}^+ &= \Delta_\tau (D_{0,\tau}^+ \Delta_\xi u_{0,\tau}^+ - G_{0,\tau}^+ u_{0,\tau} + F_{0,\tau}^+)_0, \\
\Delta_\tau^2 u_{0,0}^{\overset{1}{-}} &= \Delta_\tau (f_\tau^{\overset{1}{-}} - B_\tau^1 u_{0,\tau})_0, \\
\Delta_\tau^2 u_{0,0}^{\overset{2}{-}} &= \Delta_\tau^2 (f_\tau^{\overset{2}{-}} - B_\tau^{2+} u_{0,\tau}^+ - B_\tau^{2\overset{1}{-}} u_{0,\tau}^{\overset{1}{-}})_0,
\end{aligned}$$

and we may conclude succesively that $\Delta_\xi \Delta_\tau u_{0,0}^+$, $\Delta_\tau^2 u_{0,0}^+$, $\Delta_\tau^2 u_{0,0}^{\overset{1}{-}}$, and $\Delta_\tau^2 u_{0,0}^{\overset{2}{-}}$ remain bounded. For $\Delta_\xi \Delta_\tau u_{0,0}^{\overset{1}{-}}$ and $\Delta_\xi \Delta_\tau u_{0,0}^{\overset{2}{-}}$, after simple computations, we obtain

$$\begin{aligned}
\Delta_\xi \Delta_\tau u_{0,0}^{\overset{1}{-}} = n\{&\mathscr{D}^{\overset{1}{-}}(0,0)\, d\mathscr{f}^{0\overset{1}{-}}(0) - \mathscr{G}^{\overset{1}{-}}(0,0)\, \mathscr{f}^0(0) + \mathscr{F}^{\overset{1}{-}}(0,0) + \\
&+ \mathscr{B}^1(0)\, \mathscr{f}^0(0) - \mathscr{f}^{\overset{1}{-}}(0)\} + \mathscr{O}(1) \quad \text{when} \quad n \to \infty, \\
\Delta_\xi \Delta_\tau u_{0,0}^{\overset{2}{-}} = n\{&\mathscr{B}^2(0)\, [\mathscr{D}(0,0)\, d\mathscr{f}^0(0) - \mathscr{G}(0,0)\, \mathscr{f}^0(0) + \mathscr{F}(0,0)] + \\
&+ d\mathscr{B}^2(0)\, \mathscr{f}^0(0) - d\mathscr{f}^{\overset{2}{-}}(0) + \\
&+ \mathscr{B}^{2\overset{1}{-}}(0)\, [\mathscr{f}^{\overset{1}{-}}(0) - \mathscr{D}^{\overset{1}{-}}(0,0)\, d\mathscr{f}^{0\overset{1}{-}}(0) + \mathscr{G}^{\overset{1}{-}}(0,0)\, \mathscr{f}^0(0) - \\
&- \mathscr{F}^{\overset{1}{-}}(0,0) - \mathscr{B}^1(0)\, \mathscr{f}^0(0)]\} + \mathscr{O}(1) \quad \text{when} \quad n \to \infty.
\end{aligned}$$

According to (4.3) and (4.5) this implies the boundedness of $|u_n^{\overset{i}{-}}|_{0,2}$, $i=1, 2$, and the proof of Lemma 4.1 is completed.

Lemma 4.2. *If* $\mathscr{F}, \mathscr{f}^{\overset{1}{-}} \in \mathscr{C}^q$; $\mathscr{f}^0, \mathscr{f}^{\overset{2}{-}} \in \mathscr{C}^{q+1}$, *and if*

$$\begin{aligned}
\partial^\alpha \mathscr{F}(0,0) &= 0, & |\alpha| &\leqq q-1, \\
d^\beta \mathscr{f}^0(0) &= 0, & \beta &\leqq q, \\
d^\beta \mathscr{f}^{\overset{1}{-}}(0) &= 0, & \beta &\leqq q-1, \\
d^\beta \mathscr{f}^{\overset{2}{-}}(0) &= 0, & \beta &\leqq q,
\end{aligned} \tag{4.6}$$

then there is a constant C such that if u_n is the solution of the discrete boundary-value problem (2.1), *then for all n*

$$|u_n^-|_{0,q+1} \leqq C. \tag{4.7}$$

Proof. According to (2.2) and (4.6) we have the following expression, in which the coefficients $P_{\xi,\tau}$, Q_τ, R_ξ, and T_τ remain bounded when $n \to \infty$:

$$\begin{aligned} \Delta^\alpha u_{0,0}^- &= n^{|\alpha|-1}\Big(\sum_{\xi+\tau\leqq|\alpha|-1} P_{\xi,\tau} F_{\xi,\tau} + \sum_{\tau=0}^{|\alpha|-1} Q_\tau f_\tau^{\overset{1}{-}}\Big) + n^{|\alpha|}\Big(\sum_{\xi=0}^{|\alpha|} R_\xi f_\xi^0 + \sum_{\tau=1}^{|\alpha|} T_\tau f_\tau^{\overset{2}{-}}\Big) \\ &= \mathcal{O}(n^{|\alpha|-q-1}) = \mathcal{O}(1) \quad \textit{when } n \to \infty \ \textit{ for } |\alpha| \leqq q+1; \end{aligned}$$

thus (4.7) holds.

5. Existence and uniqueness theorems

The following uniqueness and existence theorems are proved in the same way as are the corresponding theorems in [*1*].

Theorem 5.1. *If $\mathscr{F}, f^0, f^- \in \mathscr{C}^0$, then there is at most one solution $u \in \mathscr{C}_m^1(\mathscr{V})$ of the boundary-value problem* (1.3).

Theorem 5.2. *Assume that $\mathscr{D}$, $\mathscr{G}$, $\mathscr{B}$, $\mathscr{F}$, f^0, f^- $\in \mathscr{C}^2$ and that the conditions* (4.3), (4.4), *and* (4.5) *are satisfied. Then there is a solution $u \in \mathscr{C}_m^1(\mathscr{V})$ of the boundary-value problem* (1.3), *and if u_n is the solution of the corresponding discrete boundary-value problem* (2.1), *then*

$$E\,\Delta^\alpha u_n \to \partial^\alpha u, \qquad |\alpha| \leqq 1$$

uniformly in $\mathscr{V}$ when $n \to \infty$. (E is the linear interpolation operator.)

Theorem 5.3. *Assume that $q \geqq 1$, that $\mathscr{D}$, $\mathscr{G}$, $\mathscr{B}$, $\mathscr{F}$, f^0, $f^- \in \mathscr{C}^{q+1}$ and that $\mathscr{F}$, f^0, f^- satisfy compatibility conditions of order q at the origin. Then there is a solution $u \in \mathscr{C}_m^q(\mathscr{V})$ of the boundary-value problem* (1.3).

We also have the following theorem concerning the rate of convergence:

Theorem 5.4. *Assume that $\mathscr{D}$, $\mathscr{G}$, $\mathscr{B}$, $\mathscr{F}$, f^0, f^- $\in \mathscr{C}^3$ and that the compatibility relations of order 2 are satisfied at the origin. If u and u_n are the solutions of the problems* (1.3) *and* (2.1), *then*

$$|E\,u_n - u|_{\mathscr{V}} = \mathcal{O}(n^{-1}) \quad \textit{when} \quad n \to \infty. \tag{5.1}$$

Proof. Using Theorem 3.1, we get (5.1) in the same way as in the proof of Theorem 6.5 in [*1*], if we only add the following estimate $(0 \leqq \vartheta \leqq 1)$:

$$\begin{aligned} |l^{\overset{1}{-}} u_{n,\tau} - l^{\overset{1}{-}} R\,u_\tau| &= |f^{\overset{1}{-}}(\tau k) - l^{\overset{1}{-}} R\,u_\tau| \\ &= \left|\frac{\partial u^{\overset{1}{-}}}{\partial t}(0,\tau k) + B^{\overset{1}{-}}(\tau k)\,u(0,\tau k) - \Delta_\tau u^{\overset{1}{-}}(0,\tau k) - B^{\overset{1}{-}}(\tau k)\,u(0,\tau k)\right| \\ &= \frac{k}{2}\left|\frac{\partial^2 u^{\overset{1}{-}}}{\partial t^2}(0,(\tau+\vartheta)k)\right| \leqq \frac{k}{2}|u^{\overset{1}{-}}|_{\mathscr{V},2} = \mathcal{O}(n^{-1}) \quad \text{when} \quad n \to \infty. \end{aligned}$$

6. The quasi-linear case

In this section we shall give local existence and uniqueness theorems for the quasi-linear boundary-value problem

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \mathscr{D}\frac{\partial u}{\partial x} = \mathscr{F} \quad \text{in} \quad \mathscr{V}, \qquad \mathscr{D} = \mathscr{D}(x,t,u), \qquad \mathscr{F} = \mathscr{F}(x,t,u),\\
&u = \mathscr{f}^0 \quad \text{on} \quad \mathscr{S}^0, \qquad \mathscr{f}^0 = \mathscr{f}^0(x),\\
&\frac{\partial u^{\overset{1}{-}}}{\partial t} = \mathscr{f}^{\overset{1}{-}} \quad \text{on} \quad \mathscr{S}^-, \qquad \mathscr{f}^{\overset{1}{-}} = \mathscr{f}^{\overset{1}{-}}(t,u),\\
&u^{\overset{2}{-}} = \mathscr{f}^{\overset{2}{-}} \quad \text{on} \quad \mathscr{S}^-, \qquad \mathscr{f}^{\overset{2}{-}} = \mathscr{f}^{\overset{2}{-}}(t,u^+,u^{\overset{1}{-}}).
\end{aligned}
\tag{6.1}
$$

$\mathscr{V}$, $\mathscr{S}^0$, and $\mathscr{S}^-$, with λ to be specified shortly, have the same meaning as before. About $\mathscr{D}$, $\mathscr{F}$, $\mathscr{f}^0$, $\mathscr{f}^{\overset{1}{-}}$, $\mathscr{f}^{\overset{2}{-}}$ we make the following assumptions:

1) $\mathscr{f}^0 \in \mathscr{C}^2_m(\mathscr{S}^0)$. *There are closed sets*

$$
\begin{aligned}
\Omega &= \{(x,t,u) \,|\, 0 \leqq x \leqq 1;\ 0 \leqq t \leqq \delta_1;\ |u - \mathscr{f}^0(x)| \leqq \varepsilon\},\\
\Omega^{\overset{1}{-}} &= \{(t,u) \,|\, 0 \leqq t \leqq \delta_1;\ |u - \mathscr{f}^0(0)| \leqq \varepsilon\},\\
\Omega^{\overset{2}{-}} &= \{(t,u^+,u^{\overset{1}{-}}) \,|\, 0 \leqq t \leqq \delta_1;\ |u^+ - \mathscr{f}^{0+}(0)| \leqq \varepsilon;\ |u^{\overset{1}{-}} - \mathscr{f}^{0\overset{1}{-}}(0)| \leqq \varepsilon\},
\end{aligned}
$$

such that $\mathscr{D} \in \mathscr{C}^2_{m\times m}(\Omega)$, $\mathscr{F} \in \mathscr{C}^2_m(\Omega)$, $\mathscr{f}^{\overset{1}{-}} \in \mathscr{C}^2_{m^{\overset{1}{-}}}(\Omega^{\overset{1}{-}})$, $\mathscr{f}^{\overset{2}{-}} \in \mathscr{C}^2_{m^{\overset{2}{-}}}(\Omega^{\overset{2}{-}})$. $\mathscr{D}$ *can be written*

$$
\mathscr{D} = \begin{pmatrix} \mathscr{D}^+ & 0 \\ 0 & \mathscr{D}^- \end{pmatrix},
$$

where $\mathscr{D}^+ > 0$ *and* $\mathscr{D}^- < 0$ *in* Ω.

2) *The following compatibility conditions are satisfied at the origin:*

α) $$\mathscr{f}^{\overset{1}{-}}(0,\mathscr{f}^0(0)) = \mathscr{D}^{\overset{1}{-}}(0,0,\mathscr{f}^0(0))\,\frac{d}{dx}\mathscr{f}^{0\overset{1}{-}}(0) + \mathscr{F}^{\overset{1}{-}}(0,0,\mathscr{f}^0(0)),$$

β) $$\mathscr{f}^{\overset{2}{-}}(0,\mathscr{f}^{0+}(0),\mathscr{f}^{0\overset{1}{-}}(0)) = \mathscr{f}^{0\overset{2}{-}}(0),$$

γ) $$
\begin{aligned}
&\frac{\partial}{\partial t}\mathscr{f}^{\overset{2}{-}}(0,\mathscr{f}^{0+}(0),\mathscr{f}^{0\overset{1}{-}}(0)) +\\
&\quad + \frac{\partial}{\partial u^+}\mathscr{f}^{\overset{2}{-}}(0,\mathscr{f}^{0+}(0),\mathscr{f}^{0\overset{1}{-}}(0))\left[\mathscr{D}^+(0,0,\mathscr{f}^0(0))\,\frac{d}{dx}\mathscr{f}^{0+}(0) + \mathscr{F}^+(0,0,\mathscr{f}^0(0))\right] +\\
&\quad + \frac{\partial}{\partial u^{\overset{1}{-}}}\mathscr{f}^{\overset{2}{-}}(0,f^{0+}(0),\mathscr{f}^{0\overset{1}{-}}(0))\left[\mathscr{D}^{\overset{1}{-}}(0,0,\mathscr{f}^0(0))\,\frac{d}{dx}\mathscr{f}^{0\overset{1}{-}}(0) + \mathscr{F}^{\overset{1}{-}}(0,0,\mathscr{f}^0(0))\right]\\
&= \mathscr{D}^{\overset{2}{-}}(0,0,\mathscr{f}^0(0))\,\frac{d}{dx}\mathscr{f}^{0\overset{2}{-}}(0) + \mathscr{F}^{\overset{2}{-}}(0,0,\mathscr{f}^0(0)).
\end{aligned}
$$

We assume that λ is a positive number $\leqq |\mathscr{D}|^{-1}_{\Omega}$, and we consider the discrete boundary-value problem

$$
\begin{aligned}
\Delta_\tau u^+_{\xi,\tau} - D^+_{\xi,\tau}\Delta^+_\xi u^+_{\xi,\tau} &= F^+_{\xi,\tau} \quad &&\text{on} \quad V_1,\\
\Delta_\tau u^-_{\xi,\tau} - D^-_{\xi,\tau}\Delta^-_\xi u^-_{\xi,\tau} &= F^-_{\xi,\tau} \quad &&\text{on} \quad V_1^-,\\
u_{\xi,0} &= f^0_\xi \quad &&\text{on} \quad S^0,\\
\Delta_\tau u^{\overset{1}{-}}_{0,\tau} &= f^{\overset{1}{-}}_\tau \quad &&\text{on} \quad S^{\overset{1}{-}},\\
u^{\overset{2}{-}}_{0,\tau} &= f^{\overset{2}{-}}_\tau \quad &&\text{on} \quad S^{\overset{2}{-}}.
\end{aligned}
\tag{6.2}
$$

Theorem 6.1. *There is one and only one solution $u \in \mathscr{C}_m(V)$ of the discrete boundary-value problem* (6.2).

The solution can be defined by induction over τ by means of the following scheme:

$$\begin{aligned}
&u_{\xi,0} = f^{0}_{\xi}, && \xi = 0, \ldots, n, \\
&u^{+}_{\xi,\tau+1} = \lambda D^{+}_{\xi,\tau} u^{+}_{\xi+1,\tau} + (I^{+} - \lambda D^{+}_{\xi,\tau})\, u^{+}_{\xi,\tau} + k F^{+}_{\xi,\tau}, && \xi = 0, \ldots, n-\tau-1, \\
&u^{-}_{\xi,\tau+1} = (I^{-} + \lambda D^{-}_{\xi,\tau})\, u^{-}_{\xi,\tau} - \lambda D^{-}_{\xi,\tau} u^{-}_{\xi-1,\tau} + k F^{-}_{\xi,\tau}, && \xi = 1, \ldots, n-\tau-1, \\
&u^{\overset{1}{-}}_{0,\tau+1} = u^{\overset{1}{-}}_{0,\tau} + k f^{\overset{1}{-}}_{\tau}, && \\
&u^{\overset{2}{-}}_{0,\tau+1} = f^{\overset{2}{-}}_{\tau+1}. &&
\end{aligned}$$

Theorem 6.2. *Assume that $\mathscr{D}, \mathscr{F}, \mathscr{f}^{0}, \mathscr{f}^{\overset{1}{-}}, \mathscr{f}^{\overset{2}{-}}$ satisfy* 1) *and* 2). *Then there exists a $\delta > 0$ such that the boundary-value problem* (6.1) *has a solution in $\mathscr{V}^{\delta}$.*

The proof will not be carried through. It is analogous to the corresponding one in [*1*] and consists in proving that there is a $\delta > 0$ such that, independently of n, $|u|_{V^{\delta},2} \leqq C$. The modification needed is the same as that in Section 3; *i.e.*, to obtain the estimates, u is split into $(u^{+}, u^{\overset{1}{-}})$ and $u^{\overset{2}{-}}$ and not into u^{+} and u^{-}, as was the case in [*1*].

We have the following uniqueness theorem. The corresponding theorem in [*1*] was formulated a little less strongly. Therefore we give the proof of the theorem here and observe that as a special case follows a theorem stronger than that in [*1*].

Theorem 6.3. *Assume that $u \in \mathscr{C}^1$ is a solution in $\mathscr{V}^{\delta}$ of the boundary-value problem* (6.1) *and that $\mathscr{D}, \mathscr{F} \in \mathscr{C}^1(\Omega)$, $\mathscr{f}^{\overset{1}{-}} \in \mathscr{C}^1(\Omega^{\overset{1}{-}})$, $\mathscr{f}^{\overset{2}{-}} \in \mathscr{C}^1(\Omega^{\overset{2}{-}})$ where*

$$\begin{aligned}
\Omega &= \{(x,t,v) \mid 0 \leqq x \leqq 1,\ 0 \leqq t \leqq \delta;\ |v - u(x,t)| \leqq \varepsilon\}, \\
\Omega^{\overset{1}{-}} &= \{(t,v) \mid 0 \leqq t \leqq \delta,\ |v - u(0,t)| \leqq \varepsilon\}, \\
\Omega^{\overset{2}{-}} &= \{(t, v^{+}, v^{\overset{1}{-}}) \mid 0 \leqq t \leqq \delta,\ |v^{+} - u^{+}(0,t)| \leqq \varepsilon;\ |v^{\overset{1}{-}} - u^{\overset{1}{-}}(0,t)| \leqq \varepsilon\}.
\end{aligned}$$

Assume also that $\mathscr{D}$ is a diagonal matrix of the form

$$\mathscr{D} = \begin{pmatrix} \mathscr{D}^{+} & 0 \\ 0 & \mathscr{D}^{-} \end{pmatrix}$$

where $\mathscr{D}^{+} > 0$ and $\mathscr{D}^{-} < 0$ in Ω. Assume that $v \in \mathscr{C}^1$ is also a solution of (6.1) *in $\mathscr{V}^{\delta}$. Then $u = v$ in $\mathscr{V}^{\delta}$.*

Proof. We have $u(x,0) = v(x,0) = \mathscr{f}^{0}(x)$. We put

$$\delta_1 = \sup_{\substack{u=v \text{ in } \mathscr{V}^{\eta} \\ \eta \leqq \delta}} \eta \tag{6.3}$$

and claim that $\delta_1 = \delta$. We shall prove that the assumption $\delta_1 < \delta$ is contradictory. Assume that $\delta_1 < \delta$, and let δ_2 be a number with $\delta_1 < \delta_2 \leqq \delta$ such that $|v - u| \leqq \varepsilon$

in $\mathscr{V}^{\delta_2}$. In $\mathscr{V}^{\delta_2}$, $w=u-v$ satisfies

$$\begin{aligned}
&\frac{\partial w}{\partial t}(x,t)-\mathscr{D}(x,t,u)\frac{\partial w}{\partial x}(x,t)\\
&\qquad\qquad=\mathscr{F}(x,t,u)-\mathscr{F}(x,t,v)+[\mathscr{D}(x,t,u)-\mathscr{D}(x,t,v)]\frac{\partial v}{\partial x}(x,t),\\
&\frac{\partial w^{\overset{1}{-}}}{\partial t}(0,t)=\mathscr{f}^{\overset{1}{-}}(t,u)-\mathscr{f}^{\overset{1}{-}}(t,v),\\
&\ w^{\overset{2}{-}}(0,t)=\mathscr{f}^{\overset{2}{-}}(t,u^+,u^{\overset{1}{-}})-\mathscr{f}^{\overset{2}{-}}(t,v^+,v^{\overset{1}{-}}).
\end{aligned}\tag{6.4}$$

The right-hand sides in (6.4) can be written in the forms $\mathscr{G}w$, $\mathscr{B}^1 w$, $\mathscr{B}^{2+}w^+ + \mathscr{B}^{2\overset{1}{-}}w^{\overset{1}{-}}$, respectively, with $\mathscr{G}, \mathscr{B}^1, \mathscr{B}^{2+}, \mathscr{B}^{2\overset{1}{-}}\in\mathscr{C}^0$ since *e.g.* $\mathscr{F}(x,t,u)-\mathscr{F}(x,t,v)$ can be written

$$\mathscr{F}(x,t,u)-\mathscr{F}(x,t,v)=\int_0^1\frac{\partial\mathscr{F}}{\partial u}(x,t,v+\vartheta w)\,d\vartheta\cdot w.$$

Therefore w is a solution of the linear boundary-value problem

$$\big(\mathscr{D}(u)=\mathscr{D}(x,t,u(x,t)),\ \mathscr{B}^2=(-\mathscr{B}^{2+},-\mathscr{B}^{2\overset{1}{-}},I^{\overset{2}{-}})\big)$$

$$\begin{aligned}
\frac{\partial w}{\partial t}-\mathscr{D}(u)\frac{\partial w}{\partial x}-\mathscr{G}w&=0, \quad \textit{in } \mathscr{V}^{\delta_2},\\
\varrho^0 w&=0,\\
\varrho^-\frac{\partial w^{\overset{1}{-}}}{\partial t}-\mathscr{B}^1\varrho^- w&=0, \quad \textit{on } (\mathscr{S}^-)^{\delta_2},\\
\mathscr{B}^2\varrho^- w&=0, \quad \textit{on } (\mathscr{S}^-)^{\delta_2}.
\end{aligned}$$

Because of Theorem 5.1, $w=0$ in $\mathscr{V}^{\delta_2}$, which contradicts the assumption (6.3).

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Zur Theorie der Approximationen vermittels stetiger Operatoren,
Vol. 7, p. 160–167, 1961

In unserer kürzlich erschienenen Arbeit „Zur Theorie der Approximationen vermittels stetiger Operatoren“ [*1*] wurde die Differenzierbarkeit von Operatoren aus Annahmen über die Differenzenquotienten geschlossen. Die dort gegebenen allgemeinen Formulierungen müssen noch verschärft werden, wenn die angekündigten Anwendungen [*1, 2*] auf die Potentialtheorie gefolgert werden sollen.

Der Grundgedanke der benutzten Schlußweise besteht darin, aus der Stetigkeit und Kompaktheit der Differenzenquotienten

$$R(x,h)\,g = \frac{1}{h}\{T(x+h) - T(x)\}\,g$$

die Existenz der Ableitung

$$\lim_{h\to 0} \frac{1}{h}\{T(h) - T(0)\}\,g$$

zu schließen. Dieses Prinzip benötigt die folgenden Voraussetzungen:

1. $T(x)\,g$ ist stetig für $0 \leqq x \leqq a$ $\left(\lim\limits_{x\to x_0} \|T(x)\,g - T(x_0)\,g\| = 0\right)$.

2. Es gibt eine Folge h_n mit $h_n \to 0$, so daß in $0 \leqq x_2 \leqq x \leqq x_1 \leqq a$ $R(x, h_n)\,g$ gleichmäßig bezüglich x konvergiert.

Daraus folgt [*1*] mit $\varphi(x) = \lim\limits_{h_n\to 0} R(x, h_n)\,g$ für $x_2 \leq x \leq x_1$, daß $T(x_1)\,g - T(x_2)\,g = \int\limits_{x_2}^{x_1} \varphi(x)\,dx$ ist.

Für die Ziele der Approximationstheorie ist es entscheidend, aus der Beschränktheit des Differenzenquotienten auf die Existenz einer Nullfolge im Sinne der Forderung 2 schließen zu können. Dies ist beispielsweise der Fall, wenn die zugrunde gelegten Räume reflexiv sind und die Konvergenz im schwachen Sinne verstanden wird.

Ist andererseits $T(x)$ vollstetig für $x > 0$ und besitzt $T(x)$ die Halbgruppeneigenschaft, so folgt auch für nichtreflexive Räume aus der Beschränktheit von

$$A(h)\,g = \frac{1}{h}\{T(h) - I\}\,g$$

die Existenz einer Nullfolge h_n derart, daß der Differenzenquotient

$$R(x, h_n)\,g = T(x)\,A(h_n)\,g$$

im Intervall $0 < x_2 \leq x \leq x_1$ den Voraussetzungen genügt und $T(x)$ somit für alle $x \geqq x_2$ differenzierbar ist.

Diese Schlußweise läßt sich jedoch nicht auf die Differenzierbarkeit im Nullpunkt übertragen, wenn $T(0)=I$ ist, so daß die Voraussetzung 2 nur unter weiteren Annahmen gefolgert werden kann. Dies ist beispielsweise möglich, wenn die Menge $A(h)g$ nicht nur beschränkt, sondern auch kompakt ist. Legen wir die Räume L^p $(1<p<\infty)$ zugrunde, so kann die Kompaktheit aus der Beschränktheit gefolgert werden. Im Falle nichtreflexiver Räume ist die Beschränktheit des Differenzenquotienten jedoch nicht ausreichend für die Existenz der Ableitung, wie das folgende Beispiel für den Raum der stetigen Funktionen auf dem Rande des Einheitskreises zeigt (C-Norm):

Die Funktion $f(z)=\operatorname{arc\,sin}\left(i\,\frac{1+z}{1-z}\right)$ bildet das Innere des Einheitskreises so auf den Streifen $-\frac{\pi}{2}<Re\,w<\frac{\pi}{2}$, $\operatorname{Im} w>0$ ab, daß der Punkt $z=1$ in ∞ übergeht. $e^{f(z)}$ ist daher für alle z aus dem Einheitskreis gleichmäßig beschränkt. Bilden wir

$$g(z)=(1-z)\,(1+z^2)\,e^{f(z)},$$

so folgt

$$g'(z)=e^{f(z)}\{-1+2z-3z^2+i\sqrt{2}\sqrt{1+z^2}\}$$

$g(e^{i\varphi})$ ist für alle φ stetig, insbesondere ist $g(1)=0$. Wählen wir nun $U(r,\varphi)=Re\,g(z)$, so ist $U(1,\varphi)$ stetig in φ, und es gilt $U(1,0)=0$. Für festes φ erhalten wir mit $r<r_1<1$

$$\frac{U(r,\varphi)-U(1,\varphi)}{r-1}=\frac{\partial U}{\partial r}(r_1,\varphi).$$

Wegen $(\nabla U)^2=|g'|^2$, $\frac{\partial U}{\partial r}=\mathfrak{n}\,\nabla U$ folgt daraus

$$\left|\frac{U(r,\varphi)-U(1,\varphi)}{r-1}\right|\leqq|g'|\leqq C$$

gleichmäßig für alle φ. Für $\varphi=0$ besitzt jedoch der Differenzenquotient

$$\frac{U(r,0)-U(1,0)}{r-1}=-(1+r^2)\cos\log\frac{\sqrt{2(1+r^2)}+1+r}{1-r}$$

für $r\to 1-0$ keinen Grenzwert.

Die in der voraufgegangenen Arbeit benutzten Methoden führen daher dann zu den gefolgerten Anwendungen auf das Randverhalten der Potentialfunktionen bei Benutzung der Räume C, L^1 und L^∞, wenn außer der Beschränktheit der Differenzenquotienten auch deren Kompaktheit verlangt wird.

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